

SINGULARITY CATEGORIES OF SOME 2-CY-TILTED ALGEBRAS

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ABSTRACT. We define a class of finite-dimensional Jacobian algebras, which are called (simple) polygon-tree algebras, as a generalization of cluster-tilted algebras of type \mathbb{D} . They are 2-CY-tilted algebras. Using a suitable process of mutations of quivers with potentials (which are also BB-mutations) inducing derived equivalences, and one-pointed (co)extensions which preserve singularity equivalences, we find a connected selfinjective Nakayama algebra whose stable category is equivalent to the singularity category of a simple polygon-tree algebra. Furthermore, we also give a classification of algebras of this kind up to representation type.

1. INTRODUCTION

The Fomin-Zelevinsky mutation (FZ-mutation for short) of quivers plays an important role in the theory of cluster algebras initiated in [FZ]. Motivated by this theory via [MRZ], Buan, Marsh, Reiten, Reineke and Todorov introduced cluster categories to give a perfect categorical model for cluster algebras [BMRRT]. Importantly, cluster-tilting objects in cluster categories are used to categorify clusters of the corresponding cluster algebras, and their mutations correspond to the FZ-mutation of quivers [BMR2]. This is generalized to more general Hom-finite triangulated 2-Calabi-Yau (2-CY for short) categories [IY, BIRSc]. On the other hand, in [DWZ], Derksen, Weyman and Zelevinsky studied quivers with potentials (QPs for short), that is, pairs consisting of a quiver and a special element of its (complete) path algebra, and defined the mutations of such objects, thus, they provide a new representation-theoretic interpretation for FZ-mutations of quivers.

Associated with cluster-tilting objects T in cluster categories (resp. 2-CY categories) \mathcal{C} are the endomorphism algebras $\text{End}_{\mathcal{C}}(T)$, called cluster-tilted algebras [BMR1] (resp. 2-CY-tilted algebras [Rei]). Associated with QPs (Q, W) are the Jacobian algebras (see e.g., [DWZ]). In particular, cluster-tilted algebras are 2-CY-tilted algebras, and a large class of 2-CY-tilted algebras coming from triangulated 2-CY categories associated with elements in Coxeter groups ([BIRSc]) are Jacobian algebras, including cluster-tilted algebras [BIRSm]. Furthermore, any finite-dimensional Jacobian algebras are 2-CY-tilted algebras [Am].

The mutation of cluster-tilting objects induces an operation on the associated 2-CY-tilted algebras and the mutation of QPs induces an operation on the associated Jacobian algebras. It is a conjecture that for 2-CY-tilted algebras which are Jacobian algebras, the mutations of 2-CY-tilted algebras coincides with those of Jacobian algebras. This is proved for a large class of 2-CY-tilted algebras in [BIRSm], including cluster-tilted algebras, see Theorem 2.7.

To know when a single mutation of QP leads to derived equivalence of the corresponding Jacobian algebras, Ladkani [La] defined the negative and positive mutations of algebras, which are endomorphism algebras of tilting complexes introduced and studied by Vitória in [Vi], Keller and Yang in [KY]. In fact, the negative mutation of algebras is a generalization of BB-tilting modules [BB], which are themselves generalizations of the BGP reflection functors introduced

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in [BGP]. For 2-CY-tilted algebras, Ladkani proved that the mutation of cluster-tilting objects inducing derived equivalence coincides with the BB-mutation under some conditions, see Proposition 2.14. These kind of mutations are used to give derived equivalence classification of cluster-tilted algebras of Dynkin type \mathbb{D} [BHL2] and \mathbb{E} [BHL1].

Besides, 2-CY-tilted algebras are Gorenstein algebra of dimension at most one, and their categories of Gorenstein projective modules (also called maximal Cohen-Macaulay modules) are stably 3-CY [KR]. A fundamental result of Buchweitz [Bu] and Happel [Ha] states that for a Gorenstein algebra A , the stable category of Gorenstein projective modules over A is equivalent to its singularity category $D_{sg}^b(A)$, where the singularity category of an algebra is defined to be the Verdier quotient of the bounded derived category with respect to the thick subcategory formed by complexes isomorphic to bounded complexes of finitely generated projective modules [Bu, Ha, Or]. The singularity categories of many algebras have been described clearly, see e.g., [C1, C2, Ka, CL]. In [CGL], we have settled the problem of singularity equivalence classification of the cluster-tilted algebras of type \mathbb{A} , \mathbb{D} and \mathbb{E} .

In this paper, as a generalization of a type of cluster-tilted algebras of type \mathbb{D} , we define a class of algebras, which are called polygon-tree algebras, see Definition 3.4. Roughly speaking, they are Jacobian algebras with their quivers constructed from several oriented cycles like trees, which are called polygon-tree quivers, and the potentials are primitive in the sense of [DWZ]. Locally, the quiver of the polygon-tree algebra is called a floriated quiver, see Section 3. Both polygon-tree quivers and floriated quivers are cyclically oriented quivers in the sense of [BGZ, BT].

First, with the help of [TV], we prove that the polygon-tree algebras, including floriated algebras, are finite-dimensional Jacobian algebras, and then are 2-CY-tilted algebras, see Proposition 3.8. Furthermore, we obtain that the polygon-tree algebras are schurian algebras, see Theorem 3.11. Second, we consider the singularity categories of the simple polygon-tree algebras, which form a subclass of polygon-tree algebras. Using a suitable process of mutations of QPs inducing derived equivalences (which are BB-mutations), and one-pointed (co)extensions which preserve singularity equivalences, we find a connected selfinjective Nakayama algebra whose stable category is equivalent to the singularity category of a simple polygon-tree algebra, see Theorem 4.4. In particular, simple polygon-tree algebras are CM-finite algebras. With the help of the description of the stable categories of representation finite selfinjective algebras of type \mathbb{A}_n in [Rie2], we describe the singularity categories of the simple polygon-tree algebras clearly, see Corollary 4.9. Third, using the classification of the quivers of finite mutation type [FeST], and the representation type of Jacobian algebras in [GLS], we also give a classification of representation type for polygon-tree algebras, see Theorem 5.6.

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2. PRELIMINARIES

In this paper, K is an algebraically closed field.

2.1. Singularity categories and Gorenstein algebras. Let Γ be a finite-dimensional K -algebra. Let $\text{mod } \Gamma$ be the category of finitely generated left Γ -modules. For an arbitrary Γ -module ${}_{{\Gamma}}X$, we denote by $\text{proj. dim}_{\Gamma} X$ (resp. $\text{inj. dim}_{\Gamma} X$) the projective dimension (resp. the injective dimension) of the module ${}_{{\Gamma}}X$. A Γ -module G is *Gorenstein projective*, if there is an exact sequence

$$P^{\bullet} : \cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots$$

of projective Γ -modules, which stays exact under $\text{Hom}_\Gamma(-, \Gamma)$, and such that $G \cong \text{Ker } d^0$. We denote by $\text{Gproj}(\Gamma)$ the subcategory of Gorenstein projective Γ -modules.

Definition 2.1 ([AR1, AR2, Ha]). *A finite-dimensional algebra Γ is called a Gorenstein algebra if Γ satisfies $\text{inj. dim } \Gamma_\Gamma < \infty$ and $\text{inj. dim } {}_\Gamma \Gamma < \infty$.*

Observe that for a Gorenstein algebra Γ , we have $\text{inj. dim } \Gamma_\Gamma = \text{inj. dim } {}_\Gamma \Gamma$, [Ha, Lemma 6.9]; the common value is denoted by $\text{Gd } \Gamma$. If $\text{Gd } \Gamma \leq d$, we say that Γ is d -Gorenstein.

An algebra is of *finite Cohen-Macaulay type*, or simply, *CM-finite*, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules.

For an algebra Γ , the *singularity category* of Γ is defined to be the quotient category $D_{sg}^b(\Gamma) := D^b(\Gamma)/K^b(\text{proj. } \Gamma)$ [Bu, Ha, Or]. Note that $D_{sg}^b(\Gamma)$ is zero if and only if $\text{gldim } \Gamma < \infty$ [Ha].

Theorem 2.2 ([Bu, Ha]). *Let Γ be a Gorenstein algebra. Then $\text{Gproj}(\Gamma)$ is a Frobenius category with the projective modules as the projective-injective objects. The stable category $\underline{\text{Gproj}}(\Gamma)$ is triangle equivalent to the singularity category $D_{sg}^b(\Gamma)$ of Γ .*

2.2. Quiver with potential and its mutations. We follow [DWZ] to introduce the quiver with potential and its mutation. A quiver $Q = (Q_v, Q_a, s, t)$ consists of a pair of finite sets Q_v (vertices) and Q_a (arrows) supplied with two maps $s : Q_a \rightarrow Q_v$ (source) and $t : Q_a \rightarrow Q_v$ (target). The complete path algebra \widehat{KQ} is the completion of the path algebra KQ with respect to the ideal generated by the arrows of Q . A *potential* on Q is an element of the closure $\text{Pot}(KQ)$ of the space generated by all non trivial cyclic paths of Q . Two potentials W and W' are called *cyclically equivalent* if $W - W' \in \{KQ, KQ\}$, where $\{KQ, KQ\}$ denotes the closure of the vector space spanned by commutators. Let \mathfrak{m} be the ideal of \widehat{KQ} generated by all arrows of Q . Let W be a potential on Q such that W is in \mathfrak{m}^2 and no two cyclically equivalent cyclic paths appear in the decomposition of W . Then the pair (Q, W) is called a *quiver with potential* (QP for short).

Two QPs (Q, W) and (Q', W') are *right-equivalent* if Q and Q' have the same set of vertices and there exists an algebra isomorphism $\varphi : \widehat{KQ} \rightarrow \widehat{KQ'}$ whose restriction on vertices is the identity map and $\varphi(W)$ and W' are cyclically equivalent. Such an isomorphism φ is called a *right-equivalence*.

For an arrow a of Q , the cyclic derivative $\partial_a W$ is defined by

$$\partial_a(a_1 \cdots a_l) = \sum_{a_i=a} a_{i+1} \cdots a_l a_1 \cdots a_{i-1}$$

and extended linearly and continuously. The *Jacobian algebra* of a QP (Q, W) , denoted by $J(Q, W)$, is the quotient of the complete path algebra \widehat{KQ} by the *Jacobian ideal* $J(W)$, where $J(W)$ is the closure of the ideal generated by $\partial_a W$, where a runs over all arrows of Q . It is clear that two right-equivalent QPs have isomorphic Jacobian algebras. A QP is called *reduced* if $\partial_a W$ is contained in \mathfrak{m}^2 for all arrows a of Q .

It is shown in [DWZ] that for any QP (Q, W) , there exists a reduced QP $(Q_{\text{red}}, W_{\text{red}})$ such that

$$J(Q, W) \simeq J(Q_{\text{red}}, W_{\text{red}}),$$

which is uniquely determined up to right-equivalence. We call $(Q_{\text{red}}, W_{\text{red}})$ the *reduced part* of (Q, W) .

For every QP (Q, W) , we define its *deformation space* $\text{Def}(Q, W)$ by

$$\text{Def}(Q, W) = \text{Tr}(J(Q, W))/R,$$

where $\text{Tr}(J(Q, W)) = J(Q, W)/\{J(Q, W), J(Q, W)\}$ and $R = k^{Q_v}$. We call a QP (Q, W) *rigid* if $\text{Def}(Q, W) = \{0\}$, i.e., if $\text{Tr}(J(Q, W)) = R$.

Let (Q, W) be a QP. Let i be a vertex of Q . Assume the following conditions:

- (c1) the quiver Q has no loops;
- (c2) the quiver Q does not have oriented 2-cycles at i ;
- (c3) no cyclic path occurring in the expansion of W starts and ends at i .

We define a new QP $\tilde{\mu}_i(Q, W) = (Q', W')$ as follows.

- Step 1 For each arrow β with target i and each arrow α with source i , add a new arrow $[\alpha\beta]$ from the source of β to the target of α .
- Step 2 Replace each arrow α with source or target i with an arrow α^* in the opposite direction.
- Step 3 The new potential W' is the sum of two potentials W'_1 and W'_2 . The potential W'_1 is obtained from W by replacing each composition $\alpha\beta$ by $[\alpha\beta]$, where β is an arrow with target i . The potential W'_2 is given by $W'_2 = \sum_{\alpha, \beta} [\alpha\beta] \beta^* \alpha^*$, where the sum ranges over all pairs of arrows α and β such that β ends at i and α starts at i .

We define $\mu_i(Q, W)$ as the reduced part of $\tilde{\mu}_i(Q, W)$, and call μ_i the *mutation at the vertex i* . In this case, μ_i is also well-defined on the QP $\mu_i(Q, W)$, and $\mu_i(\mu_i(Q, W))$ is right-equivalent to (Q, W) [DWZ].

Definition 2.3 ([DWZ]). Let $k_1, \dots, k_l \in Q_v$ be a finite sequence of vertices such that $k_p \neq k_{p+1}$ for $p = 1, \dots, l-1$. We say that a QP (Q, W) is (k_l, \dots, k_1) -nondegenerate if all the quivers with potentials

$$(Q, W), \mu_{k_1}(Q, W), \dots, \mu_{k_l} \cdots \mu_{k_1}(Q, W)$$

are 2-acyclic. We say that (Q, W) is nondegenerate if it is (k_l, \dots, k_1) -nondegenerate for every sequence of vertices as above.

Proposition 2.4 ([DWZ]). (a) If a reduced QP (Q, W) is 2-acyclic and rigid, then for any vertex k , $\mu_k(Q, W)$ is also rigid.

(b) Each rigid reduced QP (Q, W) is 2-acyclic.

(c) Each rigid QP is nondegenerate.

Definition 2.5 ([BGZ, BT]). A walk of length p in a quiver Q is a $(2p+1)$ -tuple

$$w = (x_p, \alpha_p, x_{p-1}, \alpha_{p-1}, \dots, x_1, \alpha_1, x_0)$$

such that for all i we have $x_i \in Q_0$, $\alpha_i \in Q_1$ and $\{s(\alpha_i), t(\alpha_i)\} = \{x_p, x_{p-1}\}$. The walk w is oriented if either $s(\alpha_i) = x_{i-1}$ and $t(\alpha_i) = x_i$ for all i or $s(\alpha_i) = x_i$ and $t(\alpha_i) = x_{i-1}$ for all i . Furthermore, w is called a cycle if $x_0 = x_p$. An oriented walk is also called a path. If w is oriented and $t(\alpha_i) = s(\alpha_{i+1})$ for any i , we omit the vertices and abbreviate w by $\alpha_p \dots \alpha_1$.

A cycle $c = (x_p, \alpha_p, \dots, x_1, \alpha_1, x_p)$ is called non-self-intersecting if its vertices x_1, \dots, x_p are pairwise distinct. A non-self-intersecting cycle of length 2 is called a 2-cycle. If c is a non-self-intersecting cycle, then any arrow $\beta \in Q \setminus \{\alpha_1, \dots, \alpha_p\}$ with $\{s(\beta), e(\beta)\} \subseteq \{x_1, \dots, x_p\}$ is called a chord of c . A cycle c is called chordless if it is non-self-intersecting and there is no chord of c .

A quiver Q without loops or oriented 2-cycles is called cyclically oriented if each chordless cycle is oriented.

A path γ , which is anti-parallel to an arrow η in a quiver Q , is called a shortest path if the full subquiver generated by the induced oriented cycle $\eta\gamma$ is chordless.

Definition 2.6 ([DWZ]). Let Q be a quiver. A primitive potential S is a linear combination of all oriented chordless cycles in Q with non-zero scalars.

2.3. Mutation of cluster-tilting objects. Let \mathcal{C} be a Hom-finite triangulated k -category. We denote by $[1]$ the shift functor in \mathcal{C} . Then \mathcal{C} is said to be n -Calabi-Yau (n -CY for simplicity) if there is a functorial isomorphism

$$D \operatorname{Hom}_{\mathcal{C}}(A, B) \simeq \operatorname{Ext}_{\mathcal{C}}^n(B, A)$$

for A, B in \mathcal{C} and $D = \operatorname{Hom}_k(-, k)$.

Let \mathcal{C} be a 2-CY triangulated category. An object in \mathcal{C} is a *cluster-tilting object* if

$$\text{add } T = \{X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^1(T, X) = 0\}$$

(see [BMRRT, KR]). In this case the algebra $\text{End}_{\mathcal{C}}(T)$ is called a *2-CY-tilted algebra* [Rei]. [Am, Corollary 3.7] shows that each finite-dimensional Jacobian algebra $J(Q, W)$ is 2-CY-tilted.

Let $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ be a cluster-tilting object, where T_i are nonisomorphic indecomposable objects. Then for any $1 \leq k \leq n$ there exists a unique T_k^* non-isomorphic to T_k such that $\mu_k(T) = (T/T_k) \oplus T_k^*$ is a cluster-tilting object. Moreover, there are so-called exchange triangles

$$T_k^* \xrightarrow{g} U_k \xrightarrow{f} T_k \rightarrow T_k^*[1] \text{ and } T_k \xrightarrow{g'} U'_k \xrightarrow{f'} T_k^* \rightarrow T_k[1],$$

where f and f' are minimal right $\text{add}(T/T_k)$ -approximations, and g and g' are minimal left $\text{add}(T/T_k)$ -approximations [BMRRT, IY].

Recall that a finite-dimensional algebra Λ satisfies the *vanishing condition* at k if

$$\text{Hom}_{\Lambda}(\text{Ext}_{\Lambda}^1(D\Lambda, S_k), S_k) = 0$$

holds for the simple Λ -module S_k .

Theorem 2.7 ([BIRSm]). *Let \mathcal{C} be a 2-CY triangulated category with a basic cluster-tilting object T . If $\text{End}_{\mathcal{C}}(T) \simeq J(Q, W)$ for a QP (Q, W) , no oriented 2-cycles start in the vertex k of Q and the vanishing condition is satisfied at k , then $\text{End}_{\mathcal{C}}(\mu_k(T)) \simeq J(\mu_k(Q, W))$.*

Cluster categories are by definition the orbit categories $\mathcal{C}_Q = D^b(KQ)/\tau^{-1}[1]$, where Q is a finite connected acyclic quiver, and τ is the AR-translation in $D^b(KQ)$ [BMRRT]. These orbit categories are triangulated categories [Ke1], and are Hom-finite 2-CY [BMRRT]. The cluster-tilted algebras are by definition the 2-CY-tilted algebras coming from cluster categories [BMR1].

2.4. Mutation of algebras. We recall the notion of mutations of algebras from [La]. Let $A = KQ/I$ be an algebra given as a quiver with relations. For any vertex i of Q , there is a trivial path e_i of length 0; the corresponding indecomposable projective module $P_i = Ae_i$ is spanned by the images of the paths starting at i . Thus an arrow $i \xrightarrow{\alpha} j$ gives rise to a map $P_j \rightarrow P_i$ given by right multiplication with α . Furthermore, for any vertex i of Q , there is an indecomposable injective module I_i , and a simple module S_i .

Let k be a vertex of Q without loops. Consider the following two complexes of projective A -modules

$$T_k^-(A) = (P_k \xrightarrow{f} \bigoplus_{j \rightarrow k} P_j) \oplus (\bigoplus_{i \neq k} P_i), \quad T_k^+(A) = (\bigoplus_{k \rightarrow j} P_j \xrightarrow{g} P_k) \oplus (\bigoplus_{i \neq k} P_i)$$

where the map f is induced by all the maps $P_k \rightarrow P_j$ corresponding to the arrows $j \rightarrow k$ ending at k , the map g is induced by the maps $P_j \rightarrow P_k$ corresponding to the arrows $k \rightarrow j$ starting at k , the term P_k lies in degree -1 in $T_k^-(A)$ and in degree 1 in $T_k^+(A)$, and all other terms are in degree 0 .

Definition 2.8 ([BHL2]). *Let A be an algebra given as a quiver with relations and k a vertex without loops.*

(a) *We say that the negative mutation of A at k is defined if $T_k^-(A)$ is a tilting complex over A . In this case, we call the algebra $\mu_k^-(A) = \text{End}_{D^b(A)}(T_k^-(A))$ the negative mutation of A at the vertex k .*

(b) *We say that the positive mutation of A at k is defined if $T_k^+(A)$ is a tilting complex over A . In this case, we call the algebra $\mu_k^+(A) = \text{End}_{D^b(A)}(T_k^+(A))$ the positive mutation of A at the vertex k .*

Let Q^{op} be the *opposite quiver* of Q . Namely, it has the same set of vertices as Q , with the (opposite) arrow $\alpha^* : j \rightarrow i$ for any arrow $\alpha : i \rightarrow j$ of Q . If $A = KQ/I$, then the *opposite algebra* A^{op} can be written as $A^{op} = KQ^{op}/I^{op}$ where I^{op} is generated by the paths opposite to those generating I . The indecomposable projective A^{op} -module corresponding to a vertex of Q^{op} is then $P_i^t = \text{Hom}_A(P_i, A)$. In particular, $(-)^t = \text{Hom}(-, A) : \text{mod } A \rightarrow A^{op}$ induces a duality between the category $\text{proj. } A$ and the category $\text{proj. } A^{op}$.

Lemma 2.9. *Let A be an algebra given as a quiver with relations and k a vertex without loops. Then $\mu_k^-(A^{op}) \simeq (\mu_k^+(A))^{op}$ and $\mu_k^+(A^{op}) \simeq (\mu_k^-(A))^{op}$.*

Proof. We only need prove $\mu_k^-(A^{op}) \simeq (\mu_k^+(A))^{op}$. It is easy to see that $(T_k^+(A))^t$ is $T_k^-(A^{op})$, where $(-)^t$ is defined for complexes of projective modules naturally. Then the result follows immediately since $(-)^t$ induces a duality between $\text{proj. } A$ and $\text{proj. } A^{op}$. \square

Proposition 2.10 ([La]). *Let k be a vertex without loops.*

(a) *The negative mutation $\mu_k^-(A)$ is defined if and only if for any non-zero linear combination $\sum a_r p_r$ of paths p_r starting at k and ending at some vertex $i \neq k$, there exists at least one arrow α ending at k such that the composition $\sum a_r p_r \alpha$ is not zero.*

(b) *The positive mutation $\mu_k^+(A)$ is defined if and only if for any non-zero linear combination $\sum a_r p_r$ of paths p_r starting at some vertex $i \neq k$ and ending at k , there exists at least one arrow β starting at k such that the combination $\sum a_r \beta p_r$ is not zero.*

(c) *$T_k^-(A)$ is a tilting complex for A if and only if $T_k^+(A^{op})$ is a tilting complex for A^{op} .*

Let τ denote the Auslander-Reiten translation in $\text{mod } A$.

Definition 2.11 ([La]). *We say that the BB-tilting module is defined at the vertex k if the A -module*

$$T_k^{BB} = \tau^{-1} S_k \oplus \left(\bigoplus_{i \neq k} P_i \right)$$

is a tilting module of projective dimension at most 1. In this case, T_k^{BB} is called the BB-tilting module associated with k .

Lemma 2.12 ([La]). *Assume that S_k is not a submodule of the radical of P_k . If T_k^- is a tilting complex, then the BB-tilting module is defined at k and $T_k^- \simeq T_k^{BB}$ in $D^b(A)$.*

Note that the above lemma holds when A is *schurian*. Recall that an algebra $A = KQ/I$ is schurian if $\dim_k \text{Hom}_A(P_i, P_j) \leq 1$ for any two vertices i, j of Q , or in other words, the entries of its Cartan matrix are only 0 or 1.

Lemma 2.13 ([La]). *Let k be a vertex of Q without loops.*

(a) *If $\mu_k^-(A)$ is defined, then $\mu_k^+(\mu_k^-(A))$ is defined and isomorphic to A .*

(b) *If $\mu_k^+(A)$ is defined, then $\mu_k^-(\mu_k^+(A))$ is defined and isomorphic to A .*

(c) *If $\mu_k^{BB}(A)$ is defined, then $\mu_k^{BB}((\mu_k^{BB}(A))^{op})$ is defined and isomorphic to A^{op} .*

Proposition 2.14 ([La]). *Let U be a cluster-tilting object in a 2-CY triangulated category \mathcal{C} and let $\Lambda = \text{End}_{\mathcal{C}}(U)$ and $\Lambda' = \text{End}_{\mathcal{C}}(\mu_k(U))$ be two neighboring 2-CY-tilted algebras. Then $\Lambda' \simeq \mu_k^{BB}(\Lambda)$ if and only if the BB-tilting modules $T_k^{BB}(\Lambda)$ and $T_k^{BB}(\Lambda^{op})$ are defined. In particular, in this case Λ and Λ' are derived equivalent.*

Lemma 2.15. *Let Q be a finite quiver without loops, k a vertex of Q . Assume that the quotient algebra $A = KQ/I$ is finite-dimensional and there is at most one arrow α ending at k . Then $\text{Ext}_A^1(I_k, S_k) = 0$.*

Proof. Since A is finite-dimensional, we get that $D \text{Ext}_A^1(I_k, S_k) \simeq \underline{\text{Hom}}_A(\tau^{-1} S_k, I_k)$.

Case (1) there does not exist any arrow ending at k . Then $S_k \simeq I_k$, and so

$$\mathrm{Ext}_A^1(I_k, S_k) = \mathrm{Ext}_A^1(I_k, I_k) = 0.$$

Case (2) there exists only one arrow $\alpha : j \rightarrow k$ ending at k . Note that $j \neq k$ since Q has no loops. Then there is an exact sequence

$$P_k \xrightarrow{f} P_j \rightarrow \tau^{-1}S_k \rightarrow 0$$

where f is induced by α . So $\tau^{-1}S_k \simeq S_j$. It is easy to see that $\mathrm{Hom}_A(S_j, I_k) = 0$ since $\mathrm{soc}(I_k) = S_k \not\simeq S_j$. So $D\mathrm{Ext}_A^1(I_k, S_k) \simeq \underline{\mathrm{Hom}}_A(S_j, I_k) = 0$. \square

Lemma 2.16. *Let \mathcal{C} be a 2-CY triangulated category with a basic cluster-tilting object T . Assume that $\mathrm{End}_{\mathcal{C}}(T) \simeq J(Q, W)$ for a QP (Q, W) where Q has no loops, no oriented 2-cycles start in the vertex k of Q , and there is at most one arrow α ending at k . If $\mathrm{End}_{\mathcal{C}}(T)$ is finite-dimensional, then $\mathrm{End}_{\mathcal{C}}(\mu_k(T)) \simeq J(\mu_k(Q, W))$.*

Proof. Set $A = \mathrm{End}_{\mathcal{C}}(T) \simeq J(Q, W)$. Since A is finite-dimensional, we get that

$$D\mathrm{Hom}_A(\mathrm{Ext}_A^1(DA, S_k), S_k) \cong DS_k \otimes_A \mathrm{Ext}_A^1(DA, S_k).$$

Note that $D(S_k)$ is the right simple module corresponding to vertex k . Let $e = e_k$ be the trivial path corresponding to k . For any $\alpha \otimes \xi \in DS_k \otimes_A \mathrm{Ext}_A^1(DA, S_k)$, where $\xi \in \mathrm{Ext}_A^1(DA, S_k)$ is represented by a short exact sequence $0 \rightarrow S_k \xrightarrow{f} M \xrightarrow{g} DA \rightarrow 0$, we get that $\alpha \otimes \xi = \alpha e \otimes \xi = \alpha \otimes e\xi$. The bimodule structure of $\mathrm{Ext}_A^1(DA, S_k)$ is induced by the bimodule structure of DA , and e acts on DA on the right inducing a morphism of left modules $\beta : DA \rightarrow DA$. Noting that $DA = \bigoplus_{i \in Q_0} DAe_i = DAe \oplus (\bigoplus_{i \neq k} DAe_i)$, we get that $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, where 1 is the identity morphism for DAe . Then $e\xi$ is constructed by a pull-back as the following diagram shows.

$$\begin{array}{ccccc} e\xi : & S_k & \xrightarrow{f'} & M' & \xrightarrow{g'} DA \\ & \parallel & & \downarrow & \downarrow \beta \\ \xi : & S_k & \xrightarrow{f} & M & \xrightarrow{g} DA \end{array}$$

Since Lemma 2.15 yields that $\mathrm{Ext}_A^1(I_k, S_k) = 0$, we get that $e\xi = 0$, and then $\alpha \otimes \xi = 0$, which implies $DS_k \otimes_A \mathrm{Ext}_A^1(DA, S_k) = 0$. So A satisfies the vanishing condition at k . By Theorem 2.7, we get that $\mathrm{End}_{\mathcal{C}}(\mu_k(T)) \simeq J(\mu_k(Q, W))$. \square

Proposition 2.17. *Let A be the Jacobian algebra $J(Q, W)$ of a QP (Q, W) , k a vertex of Q and $B = J(\mu_k(Q, W))$. Assume that A is finite-dimensional, both (Q, W) and $\mu_k(Q, W)$ have no loops or oriented 2-cycles, and Q has at most one arrow ending at k . If A and B have the property that the simple modules ${}_AS_k$ and ${}_BS_k$ are not submodules of the radicals of ${}_AP_k$ and ${}_BP_k$ respectively, then*

(a) $B = J(\mu_k(Q, W)) \simeq \mu_k^-(A)$ if and only if the two algebra mutations $\mu_k^-(A)$ and $\mu_k^+(B)$ are defined.

(b) $B = J(\mu_k(Q, W)) \simeq \mu_k^+(A)$ if and only if the two algebra mutations $\mu_k^+(A)$ and $\mu_k^-(B)$ are defined.

Proof. Since A is finite-dimensional and (Q, W) has no loops or oriented 2-cycles, [DWZ, Proposition 6.4] yields that B is finite-dimensional. [Am, Corollary 3.7] shows that A is a 2-CY-tilted algebra. Denote by \mathcal{C} the 2-CY category, and T the cluster-tilting object in \mathcal{C} such that $A \simeq \mathrm{End}_{\mathcal{C}}(T)$. Since Q has only one arrow ending at k , Lemma 2.16 shows that $B = J(\mu_k(Q, W)) \simeq \mathrm{End}_{\mathcal{C}}(\mu_k(T))$.

(a) If $B = J(\mu_k(Q, W)) \simeq \mu_k^-(A)$, then the proof of [La, Theorem 4.2] implies that $T_k^-(A)$ is a tilting complex which is isomorphic to $T_k^{BB}(A)$. So $\mu_k^-(A)$ is defined, and $J(\mu_k(Q, W)) \simeq$

$\mu_k^{BB}(A)$. Then Proposition 2.14 shows that $T_k^{BB}(B^{op})$ is a tilting complex, which yields that $T_k^-(B^{op})$ is tilting by [La, Proposition 2.8]. So $T_k^+(B)$ is a tilting complex by Proposition 2.10 (c).

Conversely, if the two algebra mutations $\mu_k^-(A)$ and $\mu_k^+(B)$ are defined, then $\mu_k^{BB}(A)$ and $\mu_k^{BB}(B^{op})$ are defined by Lemma 2.12 since A and B have the property that the simple modules ${}_AS_k$ and ${}_BS_k$ are not submodules of the radicals of ${}_AP_k$ and ${}_BP_k$ respectively. The desired result follows from Proposition 2.14 easily.

(b) We consider the opposite algebras A^{op} and B^{op} . It is easy to see that $B^{op} = J(\mu_k(Q^{op}, W^{op}))$, where W^{op} is defined naturally. If $B = J(\mu_k(Q, W)) \simeq \mu_k^+(A)$, then $B^{op} = J(\mu_k(Q^{op}, W^{op})) \simeq (\mu_k^+(A))^{op} = \mu_k^-(A^{op})$, so $\mu_k^-(A^{op})$ and $\mu_k^+(B^{op})$ are defined by (a). Then we get that $\mu_k^+(A)$ and $\mu_k^-(B)$ are defined by Proposition 2.10 (c).

Conversely, let $(Q', W') = \mu_k(Q, W)$. Then $B = J(Q', W')$ and $A = J(\mu_k(Q', W'))$. If the two algebra mutations $\mu_k^+(A)$ and $\mu_k^-(B)$ are defined, by (a), we get that $A = J(\mu_k(Q', W')) \simeq \mu_k^-(B)$. Lemma 2.13 (a) shows that $\mu_k^+(\mu_k^-(B))$ is defined and isomorphic to B since $\mu_k^-(B)$ is defined. So $B \simeq \mu_k^+(\mu_k^-(B)) \simeq \mu_k^+(A)$. \square

3. POLYGON-TREE ALGEBRAS

As the introduction says, motivated by the structure of the cluster-tilted algebras of Dynkin type \mathbb{D} , we define a class of algebras, which are called *floriated algebras*.

In this section, let Q_0, Q_1, \dots, Q_n be oriented cycles such that each Q_i has m_i ($m_i \geq 3$) vertices for each Q_i . The vertices of Q_i are denoted by $v_1^i, v_2^i, \dots, v_{m_i}^i$, such that there is an arrow from v_j^i to v_{j+1}^i for $1 \leq j \leq m_i$, where we set $m_i + 1 = 1$.

We also assume that $m_0 \geq n$. Let i_1, \dots, i_n be distinct vertices of Q_0 satisfying $i_j < i_k$ for $j < k$. Identifying $v_{i_j}^0$ with v_1^j , $v_{i_j+1}^0$ with v_2^j , and also the arrow $v_{i_j}^0 \rightarrow v_{i_j+1}^0$ with the arrow $v_1^j \rightarrow v_2^j$ for $1 \leq j \leq n$, we get a quiver Q as the following diagram shows.

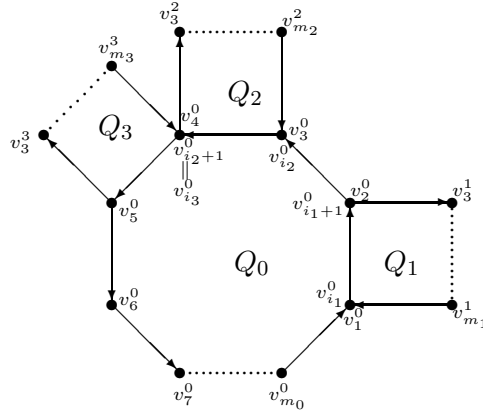


Figure 1. Floriated quiver.

$(Q_0, \{i_1, \dots, i_n\})$ (or just Q_0) is called the *central cycle*, Q_i are called the *petals* for $1 \leq i \leq n$ and $v_{i_1}^0, \dots, v_{i_n}^0$ are called *petal vertices*, Q is called the *floriated quiver of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n* (or simply *floriated quiver*).

Let $W = \sum_{j=0}^n w_j$, where w_j is the chordless oriented cycle $v_1^j \rightarrow v_2^j \rightarrow \dots \rightarrow v_{m_j}^j \rightarrow v_1^j$ along Q_j for $j = 0, 1, \dots, n$. It is easy to see that W is a primitive potential, and (Q, W) is a QP. We call the Jacobian algebra $J(Q, W)$ to be the *floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n* (or simply *floriated algebra*).

Remark 3.1. From [Va], we get that if Q_1, \dots, Q_n satisfy $m_1 = \dots = m_n = 3$, then the floriated algebra $A = J(Q, W)$ is a cluster-tilted algebra of type \mathbb{D} .

For a quiver Q , we denote by $\mathcal{S}(Q)$ the set of all the chordless cycles in Q

Proposition 3.2. Let $A = J(Q, W)$ be a floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . Then its associated QP (Q, W) is mutation equivalent to a QP (Q', W') , where Q' has only one oriented cycle c , and $W' = c$. Furthermore, (Q, W) is rigid and then nondegenerate.

Proof. If $n = 0$, then we have nothing to prove.

For $n > 0$, we assume that the quiver Q is as Figure 1 shows. Note that $m_i \geq 3$ for $1 \leq i \leq n$. Without losing generality, we assume that $m_1 = \max\{m_i | 1 \leq i \leq n\}$. If $m_1 = 3$, then $m_1 = m_2 = \dots = m_n = 3$, and so $A = J(Q, W)$ is a cluster-tilted algebra of type \mathbb{D} . From [GP, Va], we know that the mutation class of a cluster-tilted algebra of type \mathbb{D} contains an oriented cycle with the potential given by that cycle, which yields our desired result immediately. So we assume that $m_1 > 3$. We mutate the QP at the vertex $v_{m_1}^1$ and denote the resulting QP by $\mu_{v_{m_1}^1}(Q, W) = (Q^1, W^1)$. Then the quiver Q^1 is as Figure 2.1 shows and $W^1 = \sum_{w \in \mathcal{S}(Q^1)} w$.

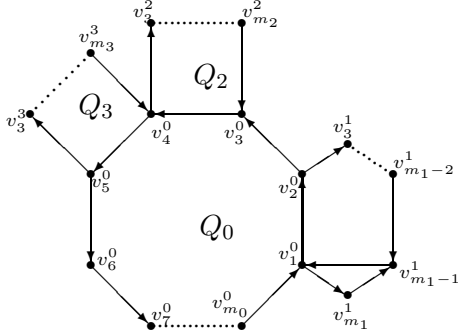


Figure 2.1. (Q^1, W^1) .

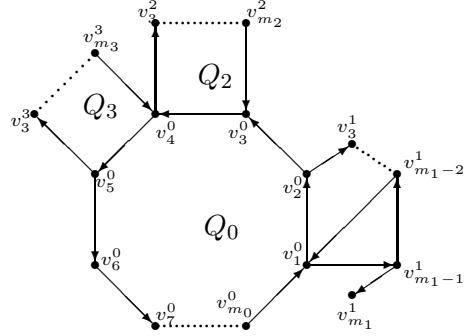


Figure 2.2. (Q^2, W^2) .

Let $(Q^2, W^2) = \mu_{v_{m_1-1}^1}(Q^1, W^1)$. Then Q^2 is as Figure 2.2 shows and $W^2 = \sum_{w \in \mathcal{S}(Q^2)} w$. Now apply $\mu_{v_{m_1-2}^1}$ to (Q^2, W^2) , and so on. After $m_1 - 3$ steps, (Q, W) is mutation equivalent to (Q^{m_1-3}, W^{m_1-3}) , where Q^{m_1-3} is as Figure 2.3 shows, and $W^{m_1-3} = \sum_{w \in \mathcal{S}(Q^{m_1-3})} w$.

Let $(Q^{m_1-2}, W^{m_1-2}) = \mu_{v_3^1}(Q^{m_1-3}, W^{m_1-3})$. Then Q^{m_1-2} is as Figure 2.4 shows, where Q_0^1 is an oriented cycle with $m_0 + 1$ vertices, and $W^{m_1-2} = \sum_{w \in \mathcal{S}(Q^{m_1-2})} w$.

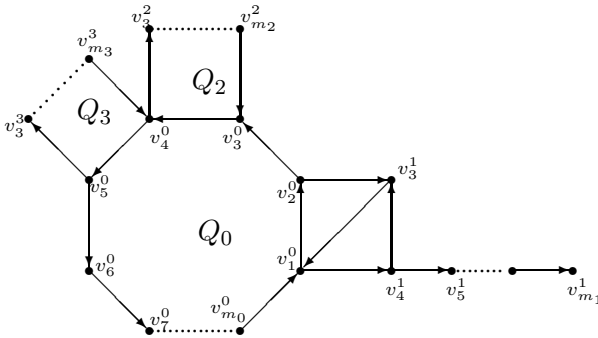


Figure 2.3. (Q^{m_1-3}, W^{m_1-3}) .

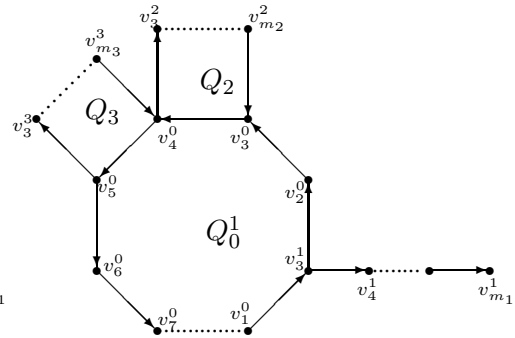


Figure 2.4. (Q^{m_1-2}, W^{m_1-2}) .

For (Q^{m_1-2}, W^{m_1-2}) , we do mutations at the vertices in Q_2 similar to the above, and inductively, we get that (Q, W) is mutation equivalent to (Q^{m-2n}, W^{m-2n}) , where $m = \sum_{i=1}^n m_i$,

and Q^{m-2n} is as Figure 3 shows, where Q_0^n is an oriented cycle with $m_0 + n$ vertices, and $W^{m-2n} = \sum_{w \in \mathcal{S}(Q^{m-2n})} w$.

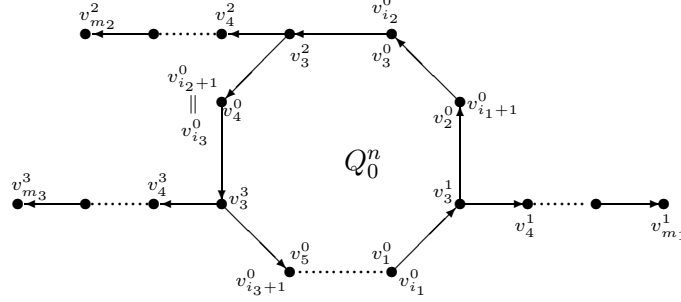


Figure 2.5. (Q^{m-2n}, W^{m-2n}) .

For (Q^{m-2n}, W^{m-2n}) , we get that $W^{m-2n} = c$ where c is the chordless cycle around Q_0^n in Q^{m-2n} . Since Q^{m-2n} has only one oriented cycle, we get that every potential on Q is cyclically equivalent to c^t for some $t > 0$, which is in $J(W^{m-2n})$, and so the proof of [DWZ, Proposition 8.1] implies that (Q^{m-2n}, W^{m-2n}) is rigid. Therefore, Proposition 2.4 yields that (Q, W) is rigid and nondegenerate. \square

Proposition 3.3. *Let $A = J(Q, W)$ be a floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . If one of the following conditions is satisfied, then A is a cluster-tilted algebra.*

- (a) *There is at most one petal with more than 3 vertices.*
- (b) *There are two petals, denoted by Q_j, Q_k for $j < k$, with more than 3 vertices, and either $k = j + 1$ and $i_{j+1} = i_j + 1$, or $k = n, j = 1$ and $i_1 + m_0 - i_n = 1$.*

Proof. First, we prove (b). We only prove for the case $j = 1, k = 2$. Proposition 3.2 implies that (Q, W) is mutation equivalent to (Q^1, W^1) with Q^1 as Figure 3.1 shows, and $W^1 = c$, where c is the non-self-intersecting oriented cycle in Q^1 .

We do mutations of QPs at $v_4^0, v_5^0, \dots, v_{m_0}^0, v_1^0, v_3^1, v_4^1, \dots, v_{m_1}^1$ successively, and then get a QP $(Q^l, W^l = 0)$, where Q^l is as Figure 3.2 shows. Since Q^l is acyclic, let \mathcal{C} be the cluster category of KQ^l , then KQ^l is a cluster-tilted algebra. Since mutations of cluster-tilted objects coincides with the mutations of QPs in this case, we get that $J(Q, W)$ is a cluster-tilted algebra.

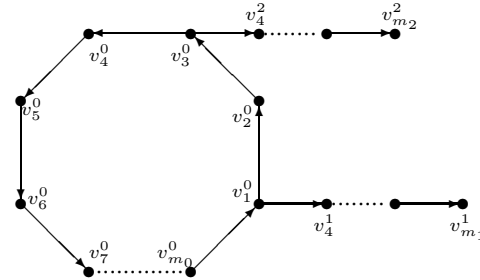


Figure 3.1. (Q^1, W^1) .

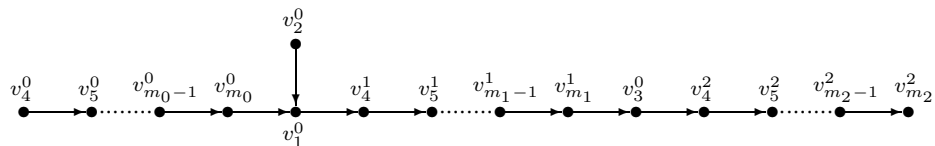


Figure 3.2. (Q^l, W^l) .

(a) Proposition 3.2 implies that (Q, W) is mutation equivalent to (Q^1, W^1) with Q^1 as the following diagram shows, and $W^1 = c$, where c is the oriented cycle in Q^1 . It is a special case of (b), we omit the proof here.

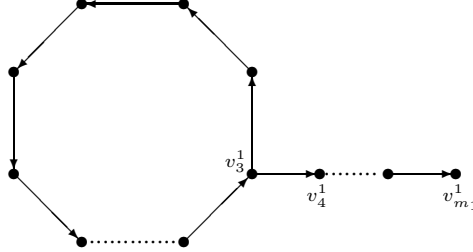


Figure 3.3. (Q^1, W^1) .

□

At the end of this section, we consider the following algebras, generalizing floriated algebras.

Definition 3.4. Let Q_0, Q_1, \dots, Q_N be oriented cycles with more than 3 vertices for each. We define a gluing quiver Q inductively.

Choose two arrows $\alpha_0 \in Q_0, \alpha_1 \in Q_1$. Define a quiver $Q_{0,1}$ from Q_0 and Q_1 by identifying α_0 and α_1 . Choose two arrows $\alpha_{0,1} \in Q_{0,1}, \alpha_2 \in Q_2$, where $\alpha_{0,1}$ is different from the glued arrow $\alpha_0 = \alpha_1$ in $Q_{0,1}$. Define a quiver $Q_{0,1,2}$ from $Q_{0,1}$ and Q_2 by identifying $\alpha_{0,1}$ and α_2 .

Inductively, choose two arrows $\alpha_{0,1,\dots,N-1} \in Q_{0,1,\dots,N-1}$ and $\alpha_N \in Q_N$, where $\alpha_{0,1,\dots,N-1}$ is different from the glued arrows $\alpha_0, \alpha_{0,1}, \dots, \alpha_{0,1,\dots,N-2}$. Define a quiver $Q = Q_{0,1,\dots,N}$ from $Q_{0,1,\dots,N-1}$ and Q_N by identifying $\alpha_{0,1,\dots,N-1}$ and α_N .

The quiver Q is called a polygon-tree quiver, and Q_0, Q_1, \dots, Q_N are called gluing components of Q . Define $W = \sum_{i=0}^N w_i$, where w_i is the non-self-intersecting oriented cycle along Q_i for $0 \leq i \leq N$. Then (Q, W) is a QP, and its Jacobian algebra $J(Q, W)$ is called a polygon-tree algebra.

Let $J(Q, W)$ be a polygon-tree algebra, where the gluing components of Q are Q_0, Q_1, \dots, Q_N . In the following, we view these gluing components as subquivers of Q naturally. We call Q_i and Q_j to be adjacent if $i \neq j$ and they have a common arrow in Q . For each Q_i , we define a full subquiver $Q(i)$ of Q consisting of Q_i and the quivers Q_j which are adjacent to Q_i . We also set that $W(i) = \sum_{w \in \mathcal{S}(Q(i))} w$.

Lemma 3.5. For any polygon-tree quiver Q with gluing components Q_0, Q_1, \dots, Q_N , the Jacobian algebra associated to $(Q(i), W(i))$ is a floriated algebra for any $0 \leq i \leq N$.

Proof. It follows from the definition of the polygon-tree algebra directly. □

In order to prove that the polygon-tree algebras are finite-dimensional, we need to give the following definitions. For a quiver Q , we denote by $\text{Path}(Q)$ the set of all the oriented paths in Q .

Definition 3.6 ([TV]). Let Q be a cyclically oriented quiver such that for any arrow α , there are at most 2 shortest paths anti-parallel to α . Let $c = \beta_L \dots \beta_1 \beta_0$ be an oriented chordless cycle. We construct a sequence of triples $(\alpha_n, \rho_n, \rho'_n) \in Q_a \times (\text{Path}(Q) \cup \{0\}) \times \text{Path}(Q)$ for each $n \in \mathbb{N} \cup \{0\}$, such that ρ_n and ρ'_n are the shortest paths anti-parallel to α_n or $\rho_n = 0$ and ρ'_n is the shortest path anti-parallel to α_n if there is only one shortest path anti-parallel to α_n , in the following way:

Step 0 We denote by α_0 the arrow β_0 and by ρ'_0 the shortest path $\beta_L \dots \beta_1$ anti-parallel to the arrow α_0 in c . If there exists other shortest path ρ_0 anti-parallel to α_0 different to ρ'_0 , then the first element of the sequence is $(\alpha_0, \rho_0, \rho'_0)$. Otherwise, the sequence is $(\alpha_n, \rho_n, \rho'_n) = (\alpha_0, 0, \rho'_0)$ for all n .

Step 1 We denote by α_1 the arrow in the path ρ_0 such that $t(\alpha_0) = s(\alpha_1)$ and by ρ'_1 the shortest path anti-parallel to α_1 in $\rho_0 \alpha_0$. If there exists other shortest path ρ_1 anti-parallel to α_1 different to ρ'_1 , then the second element of the sequence is $(\alpha_1, \rho_1, \rho'_1)$. Otherwise, $(\alpha_n, \rho_n, \rho'_n) = (\alpha_1, 0, \rho'_1)$ for each $n \geq 1$.

\vdots

Step i We denote by α_i the arrow in the path ρ_{i-1} such that $s(\alpha_{i-1}) = t(\alpha_i)$ if i is even or, $t(\alpha_{i-1}) = s(\alpha_i)$ if i is odd, and by ρ'_i the shortest path anti-parallel to α_i in $\rho_{i-1} \alpha_{i-1}$. If there exists other shortest path ρ_i anti-parallel to α_i different to ρ'_i , then the $i+1$ th element of the sequence is $(\alpha_i, \rho_i, \rho'_i)$. Otherwise, $(\alpha_n, \rho_n, \rho'_n) = (\alpha_i, 0, \rho'_i)$ for each $n \geq i$.

The sequence $\{(\alpha_n, \rho_n, \rho'_n)\}_{n \in \mathbb{N} \cup \{0\}}$ is called the cyclic sequence of c . We say that the cyclic sequence $\{(\alpha_n, \rho_n, \rho'_n)\}_{n \in \mathbb{N} \cup \{0\}}$ is finite if there exists $m \in \mathbb{N}$ such that $(\alpha_n, \rho_n, \rho'_n) = (\alpha_m, 0, \rho'_m)$ for every $n \geq m$.

Lemma 3.7 ([TV]). *Let Q be a cyclically oriented quiver such that:*

- (a) *for any arrow α , there are at most 2 shortest paths anti-parallel to α ;*
- (b) *the cyclic sequence of any oriented chordless cycle c is finite.*

If W is a primitive potential of Q , then the Jacobian algebra $J(Q, W)$ is a finite-dimensional algebra.

Proposition 3.8. *Let $A = J(Q, W)$ be a polygon-tree algebra. Then A is a finite-dimensional algebra. Moreover, A has the property that ${}_A S_i$ is not a submodule of the radical of P_i for any vertex $i \in Q$.*

Proof. Let Q_0, Q_1, \dots, Q_N be the gluing components. Assume that Q_i has m_i vertices for $0 \leq i \leq N$.

First, we prove that Q is cyclically oriented by induction on N . If $N = 0$, there is nothing to prove. We assume that $Q_{0, \dots, N-1}$ is cyclically oriented, where $Q_{0, \dots, N-1}$ is a subquiver of Q formed by Q_0, \dots, Q_{N-1} . Then Q is as the following diagram shows.

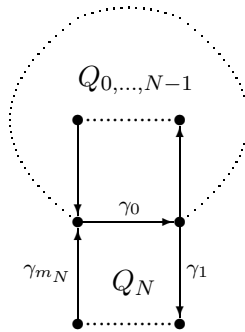


Figure 4. The quiver Q .

For any chordless cycle c of Q , if c lies in the subquiver $Q_{0, \dots, N-1}$, then c is oriented by our inductive assumption. Otherwise, if c does not lie in the subquiver $Q_{0, \dots, N-1}$, then it is easy to see that $\gamma_{m_N} \dots \gamma_1$ is a subpath of c since c is a cycle. If c is not equivalent to $\gamma_{m_N} \dots \gamma_1 \gamma_0$, then γ_0 is a chord of c , which is a contradiction. So c is equivalent to $\gamma_{m_N} \dots \gamma_1 \gamma_0$, which is oriented. Thus, Q is cyclically oriented.

Similar to the above, we also can get that the oriented chordless cycles are precisely the non-self-intersecting cycles Q_i for $0 \leq i \leq N$. So W is a primitive potential. By the construction of Q , it is easy to see that for any arrow α , there are at most 2 shortest paths anti-parallel to α .

It is also easy to get that the cyclic sequence of any oriented chordless cycle c is finite from the “tree-like” nature of the polygon-tree quiver, we omit the proof here. Then Lemma 3.7 implies that A is finite-dimensional. Furthermore, by the proof of Lemma 3.7 in [TV], any non-self-intersecting oriented cycle c is zero in A . So A has the property that ${}_A S_i$ is not a submodule of the radical of P_i for any vertex $i \in Q$. \square

Corollary 3.9. *Let A be a polygon-tree algebra. Then A is a 2-CY-tilted algebra. In particular, A is a Gorenstein algebra of dimension at most 1.*

Proof. This is obvious from [Am, Corollary 3.7] since A is a finite-dimensional Jacobian algebra. The last statement follows from [KR, Proposition 2.1] immediately, see also [KZ, Theorem 4.4]. \square

At the end of this section, we prove that all polygon-tree algebras are schurian algebras.

Definition 3.10. *Let $A = KQ/I$ be a floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . Define d_1, d_2, \dots, d_n to be*

$$d_1 = i_2 - i_1, d_2 = i_3 - i_2, \dots, d_n = i_1 + m_0 - i_n,$$

and define d_Q to be $\#\{d_i | d_i = 1, i = 1, \dots, n\}$. d_i is called the distance from Q_i to Q_{i+1} .

Theorem 3.11. *Let $A = KQ/I$ be a polygon-tree algebra. Then A is a schurian algebra.*

Proof. We denote the gluing components of the polygon-tree quiver Q by Q_0, Q_1, \dots, Q_N , and prove the statement by induction on N .

For any polygon-tree quiver with only one gluing component, the result is obvious.

We assume that the result holds for any polygon-tree quiver with no more than N gluing components.

For Q with $N + 1$ gluing components, renumbering Q_0, Q_1, \dots, Q_N , we can assume that Q_1 has only one adjacent quiver, which is denoted by Q_0 , and $Q(0)$ is the floriated quiver of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . We construct a graph $T(Q)$ from Q with vertices the gluing components Q_0, Q_1, \dots, Q_N . If Q_i and Q_j are adjacent, we add an edge between them. Then $T(Q)$ is a tree. We can view Q_0 as the root, so that the other vertices form n branches. We order the branches in such a way that the i -th branch is that containing Q_i .

The proof can be broken up into the following four cases.

Case (1): $Q(0)$ satisfies $d_1 = i_2 - i_1 > 1$ and $d_n = i_1 + m_0 - i_n > 1$. We assume that Q is as Figure 6.1 shows.

Case (2): $Q(0)$ satisfies $d_1 = i_2 - i_1 = 1$ and $d_n = i_1 + m_0 - i_n > 1$. We assume that Q is as Figure 7.1 shows.

Case (3): $Q(0)$ satisfies $d_1 = i_2 - i_1 > 1$ and $d_n = i_1 + m_0 - i_n = 1$. It follows immediately by considering Case (2) in A^{op} which is also a polygon-tree algebra.

Case (4): $Q(0)$ satisfies $d_1 = i_2 - i_1 = 1$ and $d_n = i_1 + m_0 - i_n = 1$. We assume that Q is as Figure 8.1 shows.

We give the proof only in Case (4), since the others are similar.

In Case (4), as Figure 8.1 shows, let $U = KQ'/I'$ be the polygon-tree algebra where Q' is the polygon-tree subquiver of Q with gluing components Q_0, Q_2, \dots, Q_N . For simplicity, we denote by i the vertex v_i^0 in Q_0 for each vertex v_i^0 . Then U is a schurian algebra by the inductive assumption and it is a quotient algebra of A . Let $V = KQ'/I''$ be the full subalgebra of A on the quiver Q' . Then $I' = \langle I'', \xi \rangle$, where ξ is the path $2 \xrightarrow{\gamma_2} 3 \xrightarrow{\gamma_3} \dots \xrightarrow{\gamma_{m_0-1}} m_0 \xrightarrow{\gamma_{m_0}} 1$ located in Q_0 .

Let $a, b \in Q$ be two vertices. For any path α between them in Q , if there is an oriented cycle appearing as a subpath of α , then $\alpha \in I$ by Proposition 3.8. So we only need to check that $\dim e_b A e_a \leq 1$ for $a \neq b$. The proof can be broken up into the following four cases.

Case (4a): $a, b \in Q_1$.

If $a = 1$ and $b = 2$, then there is an arrow γ_1 from 1 to 2. For any other path α from 1 to 2 different to γ_1 , since every oriented cycle is in I , we assume that α is in Q' . Because $U = KQ'/I'$ is a schurian algebra, there exist $k_1, k_2 \in K$, not all zero, such that $k_1\alpha + k_2\gamma_1 \in I'$, which yields that $\alpha \in I'$. So there exist $p_1 \in I''$ and $p_2 \in e_2 KQ'\xi KQ'e_1$ such that $\alpha = p_1 + p_2$. Obviously, $p_1 \in I$. Since p_2 passes through an oriented cycle, we get that $p_2 \in I$, and then $\alpha \in I$. Therefore, there is only one path γ_1 from 1 to 2 which is not in I , and then $\dim e_2 A e_1 = 1$.

For any path α from $a = 2$ to $b = 1$ in Q , if α is not located in Q' , then it is not hard to see that the path $\eta : 2 \rightarrow v_3^1 \rightarrow \cdots \rightarrow v_{m_1}^1 \rightarrow 1$ is a subpath of α . Since $\eta - \xi \in I$, after replacing all η with ξ in α , we can assume that α is in Q' . Because Q' is a polygon-tree quiver, we get that α passes through the vertices $3, 4, \dots, m_0$ in Q_0 , and then $\alpha = \alpha_{m_0} \cdots \alpha_3 \alpha_2$ such that $i + 1$ is the end point of α_i , and the starting point of α_{i+1} for any $2 \leq i \leq m_0 - 1$. For any $2 \leq i \leq m_0$, there is an arrow γ_i from i to $i + 1$ in Q' (where $m_0 + 1 = 1$), similar to the above, if α_i is different to γ_i , then we get that $\alpha_i \in I$, and then $\alpha \in I$. So if $\alpha \notin I$, then $\alpha = \xi$, which implies that $\dim e_1 A e_2 = 1$.

For both of $a = v_i^1$ and $b = v_j^1$ not in $\{1, 2\}$, if $3 \leq i < j \leq m_1$, then there is only one path from a to b which does not go through any oriented cycle. So $\dim e_b A e_a \leq 1$. If $3 \leq j < i \leq m_1$, then any path α from a to b passes through the vertices 1 and 2. So $\alpha = \beta_3 \beta_2 \beta_1$, where β_1 is the path $a = v_i^1 \rightarrow \cdots \rightarrow v_{m_1}^1 \rightarrow 1$, and β_3 is the path $2 \rightarrow \cdots \rightarrow v_j^1 = b$ in Q_1 . We can assume that β_2 is a path from 1 to 2 in Q' . From the above, if $\beta_2 \neq \gamma_1$, then $\beta_2 \in I$. So if $\alpha \notin I$, then $\alpha = \beta_3 \gamma_1 \beta_1$, which implies that $\dim e_b A e_a \leq 1$.

For $a = 1$ and $b = v_j^1 \notin \{1, 2\}$, then any path α from 1 to b is of form $\beta_2 \beta_1$, where β_1 is a path from 1 to 2, and β_2 is the path $2 \rightarrow v_3^1 \rightarrow \cdots \rightarrow v_j^1$ in Q_1 . From the above, we get that if $\alpha \notin I$, then $\beta_1 = \gamma_1$, so $\alpha = \beta_2 \gamma_1$, which implies that $\dim e_b A e_1 \leq 1$.

For $a = 2$ and $b = v_j^1 \notin \{1, 2\}$, there is only one path $\beta_2 = 2 \rightarrow v_3^1 \rightarrow \cdots \rightarrow v_j^1$ from $a = 2$ to b which does not pass through any oriented cycle. So $\dim e_b A e_2 \leq 1$.

For $a = v_i^1 \notin \{1, 2\}$ and $b \in \{1, 2\}$, by considering the polygon-tree algebras A^{op} , we can get that $\dim e_b A e_a \leq 1$ from the above.

Case (4b): $a = v_i^1 \in Q_1, b \in Q \setminus Q_1$. If $a \neq 2$, then any path α in Q from $a = v_i^1$ to b is of form $a = v_i^1 \rightarrow v_{i+1}^1 \rightarrow \cdots \rightarrow v_{m_1}^1 \rightarrow 1 \rightarrow \cdots \rightarrow b$. Denote by β the path $a = v_i^1 \rightarrow v_{i+1}^1 \rightarrow \cdots \rightarrow v_{m_1}^1 \rightarrow 1$.

For any two paths α_1, α_2 from a to b in Q , there are two paths δ_1, δ_2 from 1 to b in Q such that $\alpha_i = \delta_i \beta$. We need to check that α_1, α_2 are linearly dependent in A . Since every oriented cycle is in I , we only need prove for the case: δ_1 and δ_2 are in Q' . Because $U = KQ'/I'$ is a schurian algebra, there exist $k_1, k_2 \in K$, not all zero, such that $k_1\delta_1 + k_2\delta_2 \in I'$. Since $I' = \langle I'', \xi \rangle$, there exist $p_1 \in I''$ and $p_2 \in e_b KQ'\xi KQ'e_1$ such that $k_1\delta_1 + k_2\delta_2 = p_1 + p_2$, and then $k_1\alpha_1 + k_2\alpha_2 = p_1\beta + p_2\beta$. It is easy to see that $p_1\beta \in I$. Since each oriented cycle is in I , we get that $p_2\beta \in I$ and then $k_1\alpha_1 + k_2\alpha_2 \in I$. Therefore, $\dim e_b A e_a \leq 1$.

If $a = 2$, for any two paths α_1, α_2 from 2 to b , then we need to check that they are linearly dependent in A . We also only need to prove it for the case: α_1 and α_2 are in Q' , and both of them do not go through any oriented cycles. Since U is a schurian algebra, we get that there exist $k_1, k_2 \in K$, not all zero, such that $k_1\alpha_1 + k_2\alpha_2 \in I'$. Since $I' = \langle I'', \xi \rangle$, there exist $q_1 \in I''$ and $q_2 \in e_b KQ'\xi KQ'e_2$ such that $k_1\alpha_1 + k_2\alpha_2 = q_1 + q_2$. Obviously, $q_1 \in I$.

If $q_2 \in I''$, then α_1 and α_2 are linearly dependent in V and then in A .

If $q_2 \notin I''$, then there exists at least one path $l = l_2 \xi l_1$ from $a = 2$ to b in Q' , where l_1 is of the form $2 \rightarrow \cdots \rightarrow 2$, and l_2 is of the form $1 \rightarrow \cdots \rightarrow b$, such that $l \notin I''$. So $l_1 = e_2$. Note

that l_2 does not pass through γ_1 . In fact, otherwise, l passes through an oriented cycle, and then $l \in I''$, contradicts. If b is not located in an oriented cycle lying in the n -th branch, then the subpath l_2 passes through the vertex m_0 since Q is a polygon-tree quiver, which yields that $l \in I''$ since every oriented cycle in Q' is in I'' , contradicts. So b is located in an oriented cycle lying in the n -th branch and $b \neq m_0$.

Since α_1 is a path from 2 to b in Q' and Q' is a polygon-tree quiver, α_1 passes through the vertex m_0 . Then $\alpha_1 = \beta_2\beta_1$, where β_1 is a path from 2 to m_0 , β_2 is a path from m_0 to b . Since β_1 and $\gamma_{m_0-1} \cdots \gamma_2$ are two paths from 2 to m_0 in Q' , they are linearly dependent in U . Similar to the discussion in the above, we get that β_1 and $\gamma_{m_0-1} \cdots \gamma_2$ are linearly dependent in V . Since β_2 and the subpath of l : $l_2\gamma_{m_0}$ are two paths from m_0 to b in Q' , we get that they are linearly dependent in U . Similar to the discussion in the above, we can get that they are linearly dependent in V . Therefore, α_1 and l are linearly dependent in A . Similarly, α_2 and l are linearly dependent in A , and then α_1 and α_2 are linearly dependent in A .

Case (4c): $a \in Q \setminus Q_1, b = v_i^1 \in Q_1$. It follows immediately by considering Case (4b) in the polygon-tree algebra A^{op} .

Case (4d): $a, b \in Q \setminus Q_1$. For any path α from a to b in Q , similarly, we can assume that α is in Q' .

For any two paths α_1, α_2 from a to b , we need to check that they are linearly dependent in A . We also only need prove for the case : α_1 and α_2 are in Q' which do not go through any oriented cycles. Since U is a schurian algebra, we get that there exist $k_1, k_2 \in K$, not all zero, such that $k_1\alpha_1 + k_2\alpha_2 \in I'$. Since $I' = \langle I'', \xi \rangle$, there exist $p_1 \in I''$ and $p_2 \in e_b K Q' \xi K Q' e_a$ such that $k_1\alpha_1 + k_2\alpha_2 = p_1 + p_2$. Obviously, $p_1 \in I$.

If $p_2 \in I''$, then α_1 and α_2 are linearly dependent in V and then in A .

If $p_2 \notin I''$, then there exists at least one path $l = l_2\xi l_1$ from a to b in Q' , where l_1 is of form $a \rightarrow \cdots \rightarrow 2$, and l_2 is of form $1 \rightarrow \cdots \rightarrow b$, such that $l \notin I''$. Note that both of l_1 and l_2 do not pass through γ_1 . If a is not located in an oriented cycle lying in the second branch, then the subpath l_1 passes through the vertex 3 since Q is a polygon-tree quiver, which yields that $l \in I''$ since every oriented cycle in Q' is in I'' , contradicts. So a is located in an oriented cycle lying in the second branch and $a \neq 3$. Similarly, b is located in an oriented cycle lying in the n -th branch and $b \neq m_0$.

Since a is located in an oriented cycle lying in the second branch and b is located in an oriented cycle lying in the n -th branch, we get that α_1 goes through the vertices 3 and m_0 since Q' is a polygon-tree quiver. So we assume that $\alpha_1 = \beta_3\beta_2\beta_1$ with β_1 from a to 3, β_2 from 3 to m_0 and β_3 from m_0 to b . Similar to the above, we can get that β_1 and $\gamma_2 l_1$, β_2 and $\gamma_{m_0-1} \cdots \gamma_2$, β_3 and $l_2\gamma_{m_0}$ are three linearly dependent collections in V , and so α_1 and l are linearly dependent in A .

Similarly, α_2 and l are linearly dependent in V , which implies that α_1 and α_2 are linearly dependent in A . So $\dim e_b A e_a \leq 1$.

To sum up, we get that $\dim e_b A e_a \leq 1$ for any two vertices a, b in Q , which means A is a schurian algebra. \square

4. SINGULARITY CATEGORIES OF POLYGON-TREE ALGEBRAS

In order to prove the main result of this section, we describe a construction of matrix algebras which is obtained by X-W. Chen in [C2], see also [KN, Section 4]. Let A be a finite-dimensional algebra over a field K . Let ${}_A M$ and N_A be a left and right A -module, respectively. Then $M \otimes_K N$ becomes an A - A -bimodule. Consider an A - A -bimodule monomorphism $\phi : M \otimes_K N \rightarrow A$ such that ϕ vanishes both on M and N . Note that $\text{Im } \phi \subseteq A$ is an ideal. The vector space

$\Gamma = \begin{pmatrix} A & M \\ N & K \end{pmatrix}$ becomes an associative algebra via the multiplication

$$\begin{pmatrix} a & m \\ n & \lambda \end{pmatrix} \begin{pmatrix} a' & m' \\ n' & \lambda' \end{pmatrix} = \begin{pmatrix} aa' + \phi(m \otimes n) & am' + \lambda'm \\ na' + \lambda n' & \lambda\lambda' \end{pmatrix}.$$

Proposition 4.1 ([C2]). *Keep the notation and assumptions as above. Then there is a triangle equivalence $D_{sg}^b(\Gamma) \simeq D_{sg}^b(A/\text{Im } \phi)$.*

Note that the above construction contains the one-point extension and one-point coextension of algebras, where M or N is zero, see also [C1].

Definition 4.2. *Let $A = KQ/I$ be a polygon-tree algebra, where the gluing components of Q are Q_0, Q_1, \dots, Q_N . If Q does not admit any polygon-tree subquiver (with four gluing components) of one of the two forms in Figure 5, then A is called a simple polygon-tree algebra. In this case, Q is called a simple polygon-tree quiver.*

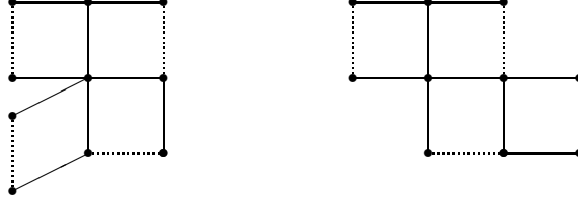


Figure 5. The banned subquivers: the edges may be oriented in any way such that the resulting quiver is cyclically oriented.

For schurian algebras, including polygon-tree algebra, it is a little easier to check that the negative and positive mutations are defined. In fact, all the algebras appearing in the Lemmas 4.5-4.8 are schurian algebras.

Lemma 4.3 ([La, BHL2]). *Let Γ be a schurian algebra.*

(a) *The negative mutation $\mu_k^-(\Gamma)$ is defined if and only if for any non-zero path $k \rightsquigarrow i$ starting at k and ending at some vertex i , there exists an arrow $j \rightarrow k$ such that the composition $j \rightarrow k \rightsquigarrow i$ is non-zero.*

(b) *The positive mutation $\mu_k^+(\Gamma)$ is defined if and only if for any non-zero path $i \rightsquigarrow k$ starting at some vertex i and ending at k , there exists an arrow $k \rightarrow j$ such that the composition $i \rightsquigarrow k \rightarrow j$ is non-zero.*

Conventions For any polygon-tree quiver Q with gluing components Q_0, Q_1, \dots, Q_N , we assume that Q_i has m_i vertices for any $0 \leq i \leq N$. Set $m = \sum_{i=0}^N m_i$, and $d_Q = \sum_{i=0}^N d_{Q(i)}$, where $d_{Q(i)}$ is defined since $Q(i)$ is a floriated quiver. It is easy to see that there is no confusion if Q is a floriated quiver.

In the following, we denote by \mathcal{N}_d the self-injective Nakayama algebra given by the path algebra of a cyclic quiver with d vertices modulo the ideal generated by paths of length $d - 1$ for any $d \geq 3$.

Theorem 4.4. *Let $A = KQ/I$ be a simple polygon-tree algebra, where the gluing components of Q are Q_0, Q_1, \dots, Q_N . Then*

$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q}).$$

In particular, A is CM-finite.

Proof. We prove the statement by induction on N .

For any simple polygon-tree algebra with only one gluing component, the result is obvious.

We assume that the result holds for any simple polygon-tree algebra with no more than N gluing components.

For Q with $N + 1$ gluing components, renumbering Q_0, Q_1, \dots, Q_N , we can assume that Q_1 has only one adjacent quiver, which is denoted by Q_0 , and $Q(0)$ is the floriated quiver of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n .

The proof can be broken up into the following four cases.

Case (1): $Q(0)$ satisfies $d_1 = i_2 - i_1 > 1$ and $d_n = i_1 + m_0 - i_n > 1$. We prove it in Lemma 4.5.

Case (2): $Q(0)$ satisfies $d_1 = i_2 - i_1 = 1$ and $d_n = i_1 + m_0 - i_n > 1$. We prove it in Lemma 4.6.

Case (3): $Q(0)$ satisfies $d_1 = i_2 - i_1 > 1$ and $d_n = i_1 + m_0 - i_n = 1$. We prove it in Lemma 4.7.

Case (4): $Q(0)$ satisfies $d_1 = i_2 - i_1 = 1$ and $d_n = i_1 + m_0 - i_n = 1$. We prove it in Lemma 4.8.

In conclusion, we get that

$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q}).$$

□

Lemma 4.5. *Keep the notation as in the proof of Theorem 4.4. If $Q(0)$ satisfies $d_1 = i_2 - i_1 > 1$ and $d_n = i_1 + m_0 - i_n > 1$, then*

$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q}).$$

Proof. Without loss of generality, we can assume that Q is as Figure 6.1 shows.

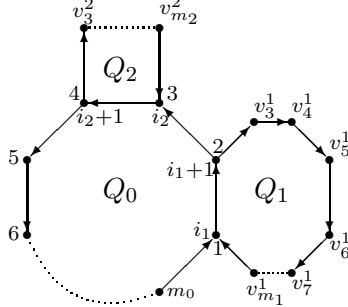


Figure 6.1. Quiver of A

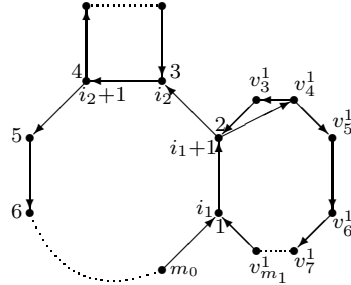


Figure 6.2. (Q^1, W^1)

Denote the QP of A by (Q, W) .

If $m_1 = 3$, let $e_{v_3^1}$ be the idempotent corresponding to v_3^1 , and $V = A/Ae_{v_3^1}A$. Similar to the proof of [CGL, Lemma 4.4] by using Proposition 4.1, we can prove that $D_{sg}^b(A) \simeq D_{sg}^b(V)$. It is easy to see that the quiver Q' of V is obtained from Q by removing the vertex v_3^1 and its adjacent two arrows. Note that V is a polygon-tree algebra with N gluing components, and $d_{Q'} = d_Q$, we get that

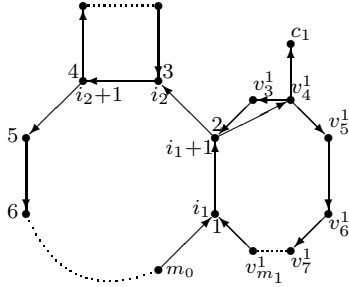
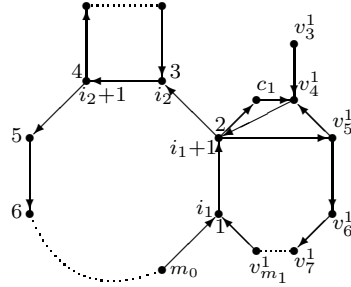
$$D_{sg}^b(V) \simeq \underline{\text{mod}}(\mathcal{N}_{\sum_{i \neq 1} m_i - 3(N-1) + d_{Q'}}) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q})$$

by the assumption of induction in the proof of Theorem 4.4 and $m_1 = 3$.

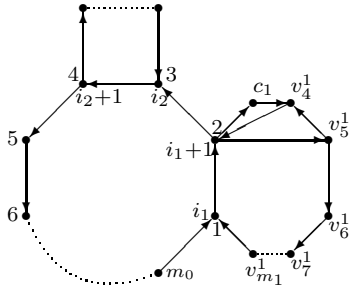
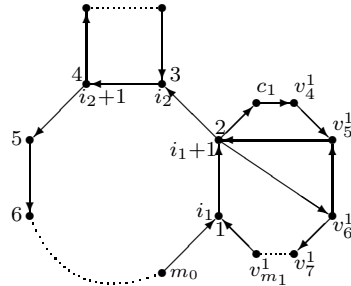
If $m_1 > 4$, it is routine to check that the vertex v_3^1 of Q satisfies Lemma 4.3 (a), which implies that the negative mutation of A at v_3^1 is defined. Doing mutation of QP at v_3^1 , we get that $\mu_{v_3^1}(Q, W) = (Q^1, W^1)$, where Q^1 is as Figure 6.2 shows, and $W^1 = \sum_{w \in \mathcal{S}(Q^1)} w$. Denote by B_1 the Jacobian algebra of (Q^1, W^1) , which is also a polygon-tree algebra. It is routine to check that the vertex v_3^1 of Q^1 satisfies Lemma 4.3 (b), which implies that the positive mutation of B_1 at v_3^1 is defined. Note that Q, Q^1 have no loops or oriented 2-cycles, and both of them are polygon-tree

algebras. Proposition 3.8 and Proposition 2.17 (a) yield that A is derived equivalent, and then singularity equivalent to B_1 .

Q^2 is constructed from Q^1 by one-pointed coextension by adding a vertex c_1 , and $W^2 = \sum_{w \in \mathcal{S}(Q^2)} w$. Denote by B_2 the Jacobian algebra of (Q^2, W^2) . Then Proposition 4.1 implies that $D_{sg}^b(B_1) \simeq D_{sg}^b(B_2)$, see also [C1, Theorem 4.1]. Easily, the vertex v_4^1 of Q^2 satisfies Proposition 2.10 (b), which implies that the positive mutation of B_2 at v_4^1 is defined. Doing mutation at v_4^1 , we get that $\mu_{v_4^1}(Q^2, W^2) = (Q^3, W^3)$, where Q^3 is as Figure 6.4 shows, and $W^3 = \sum_{w \in \mathcal{S}(Q^2)} w$. Denote by B_3 the Jacobian algebra of (Q^3, W^3) .

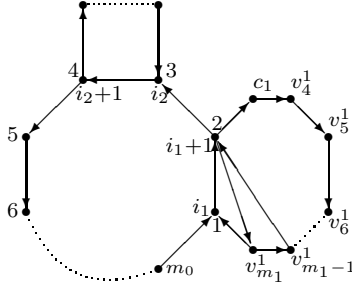
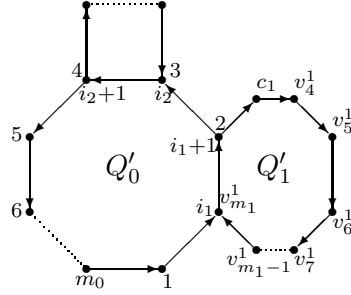
Figure 6.3. (Q^2, W^2) Figure 6.4. (Q^3, W^3)

It is routine to check that v_4^1 of Q^3 satisfies Proposition 2.10 (a), which implies that the negative mutation of B_3 at v_4^1 is defined. Note that Q^2, Q^3 have no loops or oriented 2-cycles. It is easy to see that B_2 and B_3 have the property that the simple B_2 -module $B_2 S_i$ and the simple B_3 -module $B_3 S_i$ are not submodules of the radicals of the indecomposable projective modules $B_2 P_i$ and $B_3 P_i$ for any vertex i in Q^2 and Q^3 respectively. Thus Proposition 2.17 (b) implies that B_2 is derived equivalent, and then singularity equivalent to B_3 .

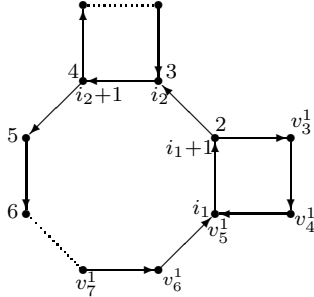
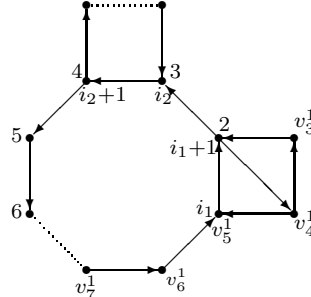
Figure 6.5. (Q^4, W^4) Figure 6.6. (Q^5, W^5)

Let Q^4 be as Figure 6.5 shows, and $W^4 = \sum_{w \in \mathcal{S}(Q^4)} w$. Denote by B_4 the Jacobian algebra of (Q^4, W^4) . Then Proposition 4.1 shows that $D_{sg}^b(B_3) \simeq D_{sg}^b(B_4)$, see also [C1, Theorem 4.1]. Lemma 4.3 (b) implies that the positive mutation of B_4 at v_5^1 is defined. Do mutation at v_5^1 , and denote $\mu_{v_5^1}(Q^4, W^4) = (Q^5, W^5)$, where Q^5 is as Figure 6.6 shows, and $W^5 = \sum_{w \in \mathcal{S}(Q^5)} w$. Denote by B_5 the Jacobian algebra of (Q^5, W^5) . Similar to the above, we get that B_4 is derived equivalent and then singularity equivalent to B_5 .

For (Q^5, W^5) , it is easy to see that the positive mutation at v_6^1 is defined, we take mutation of QP at v_6^1 , and denote the resulting QP by (Q^6, W^6) , and its Jacobian algebra by B_6 . Then take mutation at v_7^1 and inductively, after taking mutation at $v_{m_1-1}^1$, we get the Jacobian algebra B_{m_1-1} of (Q^{m_1-1}, W^{m_1-1}) with Q^{m_1-1} as Figure 6.7 shows, and $W^{m_1-1} = \sum_{w \in \mathcal{S}(Q^{m_1-1})} w$. So B_6 is derived equivalent and then singularity equivalent to B_{m_1-1} .

Figure 6.7. (Q^{m_1-1}, W^{m_1-1}) Figure 6.8. (Q^{m_1}, W^{m_1})

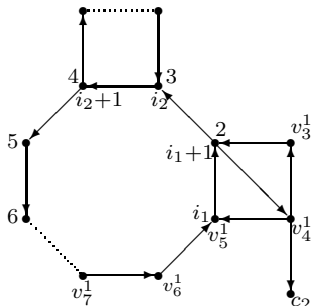
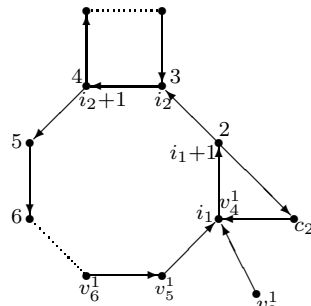
Then the positive mutation of B_{m_1-1} at $v_{m_1}^1$ is defined. Take mutation at $v_{m_1}^1$, let $(Q^{m_1}, W^{m_1}) = \mu_{v_{m_1}^1}(Q^{m_1-1}, W^{m_1-1})$, where Q^{m_1} is as Figure 6.8 shows, and $W^{m_1} = \sum_{w \in \mathcal{S}(Q^{m_1})} w$. Denote by B_{m_1} the Jacobian algebra associated to (Q^{m_1}, W^{m_1}) . Similarly, B_{m_1-1} is derived equivalent and then singularity equivalent to B_{m_1} . For convenience, we also denote the vertex c_1 in Q^{m_1} by v_3^1 . B_{m_1} is a polygon-tree algebra, denote by Q'_0, Q'_1 the oriented chordless cycles as in Figure 6.8. Then Q'_0 has $m_0 + 1$ vertices and Q'_1 has $m_1 - 1$ vertices. We do the same mutations at vertices in Q'_1 for (Q^{m_1}, W^{m_1}) as above, and then obtain a QP (Q^l, W^l) , where Q^l is as Figure 6.9 shows, $W^l = \sum_{w \in \mathcal{S}(Q^l)} w$. Denote by B_l its Jacobian algebra. Then B_{m_1} is singularity equivalent to B_l .

Figure 6.9. (Q^l, W^l) Figure 6.10. (Q^{l+1}, W^{l+1})

For (Q^l, W^l) , it is easy to see that the negative mutation at v_3^1 is defined. Take mutation at v_3^1 , denote the resulting QP by (Q^{l+1}, W^{l+1}) , and its Jacobian algebra by B_{l+1} . Then Q^{l+1} is as Figure 6.10 shows, and $W^{l+1} = \sum_{w \in \mathcal{S}(Q^{l+1})} w$. Similar to the above, we get that $D_{sg}^b(B_l) \simeq D_{sg}^b(B_{l+1})$.

Let B_{l+2} be the one-point coextension algebra of B_{l+1} , which is also a Jacobian algebra with its QP (Q^{l+2}, W^{l+2}) , where Q^{l+2} is as Figure 6.11 shows, and $W^{l+2} = \sum_{w \in \mathcal{S}(Q^{l+2})} w$. Then Proposition 4.1 implies that $D_{sg}^b(B_{l+1}) \simeq D_{sg}^b(B_{l+2})$.

It is easy to see that the negative mutation at v_4^1 is defined, we take mutation at v_4^1 , and denote the resulting algebra by B_{l+3} , which is also a Jacobian algebra with its QP (Q^{l+3}, W^{l+3}) . Then $D_{sg}^b(B_{l+2}) \simeq D_{sg}^b(B_{l+3})$.

Figure 6.11. (Q^{l+2}, W^{l+2}) Figure 6.12. (Q^{l+3}, W^{l+3})

For B_{l+3} , using Proposition 4.1 twice, we can get that $D_{sg}^b(B_{l+3}) \simeq D_{sg}^b(B_{l+4})$, where B_{l+4} is a simple polygon-tree algebra with its quiver Q^{l+4} as Figure 6.13 shows. Note that B_{l+4} has N gluing components Q_0'', Q_2, \dots, Q_N , and the oriented chordless cycle Q_0'' has $m_0 + m_1 - 3$ vertices, the other gluing components are the same as in Q . It is easy to see that $d_{Q^{l+4}} = d_Q$, so we get that

$$D_{sg}^b(A) \simeq D_{sg}^b(B_{l+4}) \simeq \underline{\text{mod}}(\mathcal{N}_{m_0+m_1-3+\sum_{i=2}^N m_i-3(N-1)+d_{Q'}}) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q})$$

by the inductive assumption in the proof of Theorem 4.4.

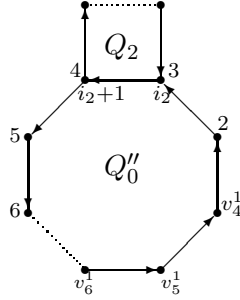


Figure 6.13. (Q^{l+4}, W^{l+4})

If $m_1 = 4$, then Q looks like the quiver as Figure 6.9 shows, after doing the same operations, we can get our desired result. \square

Lemma 4.6. *Keep the notation as in the proof of Theorem 4.4. If $Q(0)$ satisfies $d_1 = i_2 - i_1 = 1$ and $d_n = i_1 + m_0 - i_n > 1$, then*

$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q}).$$

Proof. Without loss of generality, we assume that the quiver Q is as Figure 7.1 shows.

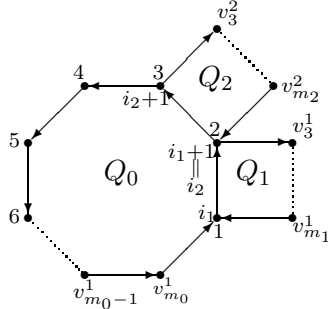


Figure 7.1. (Q, W)

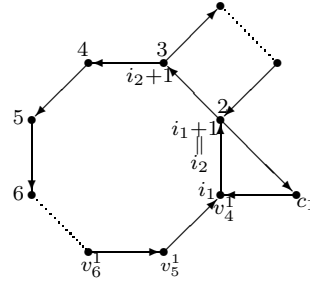


Figure 7.2. (Q^1, W^1)

If $m_1 > 3$, doing the same mutations as in Lemma 4.5, we can get that A is singularity equivalent to B_1 , where B_1 is a simple polygon-tree algebra with its QP (Q^1, W^1) as Figure 7.2 shows. For (Q^1, W^1) , the negative mutation of B_1 at c_1 is defined, so we do this mutation and get a polygon-tree algebra B_2 with its QP (Q^2, W^2) as Figure 7.3 shows. The positive mutation of B_2 at c_1 is defined, so Proposition 2.17 implies that B_1 is derived equivalent to B_2 .

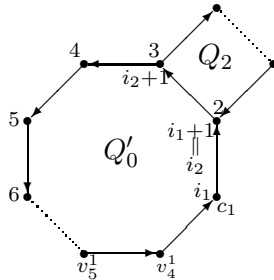


Figure 7.3. (Q^2, W^2)

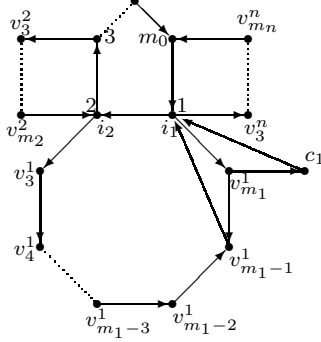
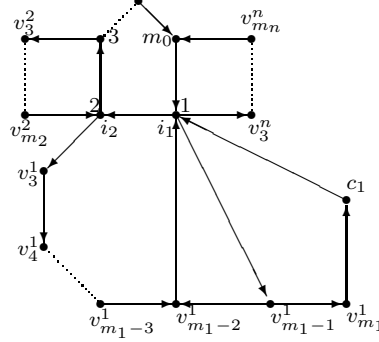
$$D_{sg}^b(A) \simeq D_{sg}^b(B_2) \simeq \underline{\text{mod}}(\mathcal{N}_{m_0+m_1-2+\sum_{i=2}^N m_i-3(N-1)+d_{Q^2}}) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q})$$

If $m_3 = 3$, then the quiver Q looks like Q^1 as Figure 7.2 shows, so we can get the desired result by doing similar mutations. \square

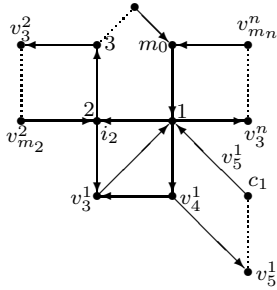
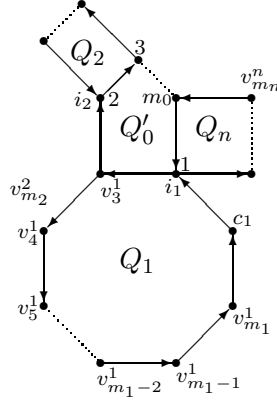
$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q}).$$
$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q}).$$

The negative mutation of B_2 at $v_{m_1-1}^1$ is defined, and we take the mutation of QP at $v_{m_1-1}^1$, denoted by (Q^3, W^3) the resulting QP, where Q^3 is as Figure 8.4 shows, and $W^3 = \sum_{w \in S(Q^3)} w$. The Jacobian algebra associated to (Q^3, W^3) is denoted by B_3 . Easily, the positive mutation of B_3 at $v_{m_1-1}^1$ is defined, so Proposition 2.17 (a) shows that B_2 is derived equivalent to B_3 .

For (Q^3, W^3) , the negative mutation at $v_{m_1-2}^1$ is defined, we take the mutation of QP at this vertex, and denote the resulting QP by (Q^4, W^4) , and its Jacobian algebra by B_4 . Similar to the above, we can get that B_3 is derived equivalent to B_4 . For (Q^4, W^4) , the negative mutation at $v_{m_1-3}^1$ is defined, we take the mutation and recursively, we get that B_4 is derived equivalent to the Jacobian algebra B_l of (Q^l, W^l) , where Q^l is Figure 8.5 shows, and $W^l = \sum_{w \in \mathcal{S}(Q^l)} w$.

Figure 8.3. (Q^2, W^2) Figure 8.4. (Q^3, W^3)

For (Q^l, W^l) , the negative mutation at v_3^1 is defined, we take the mutation of QP at it, and denote the resulting QP by (Q^{l+1}, W^{l+1}) , where Q^{l+1} is as Figure 8.6 shows, and $W^{l+1} = \sum_{w \in \mathcal{S}(Q^{l+1})} w$. Its Jacobian algebra is denoted by B_{l+1} . Similar to the above, we can get that B_l is derived equivalent to B_{l+1} .

Figure 8.5. (Q^l, W^l) Figure 8.6. (Q^{l+1}, W^{l+1})

Note that B_{l+1} is a simple polygon-tree algebra with $N+1$ gluing components Q'_0, Q_1, \dots, Q_N , and Q'_0 has $m_0 + 1$ vertices. Note that $d_{Q^{l+1}} = d_Q - 1$. So Lemma 4.6 implies that

$$D_{sg}^b(A) \simeq D_{sg}^b(B_{l+1}) \simeq \underline{\text{mod}} \mathcal{N}_{m_0+1+\sum_{i=1}^N m_i-3N+d_{Q^{l+1}}} = \underline{\text{mod}} \mathcal{N}_{m-3N+d_Q}.$$

For $m_1 = 3$, we also consider (Q^1, W^1) . The negative mutation of B_1 at v_3^1 is defined, and if we take the mutation of QP at v_3^1 , then we already get a QP as Figure 8.6 shows. It is easy to get the desired result in this case. \square

To describe the singularity category of a simple polygon-tree algebra, we need to recall a construction of Riedtmann [Rie1]. Let $\mathbb{Z}\mathbb{A}_n$ be the *translation quiver* of a quiver of type \mathbb{A}_n . The translation of $\mathbb{Z}\mathbb{A}_n$ is denoted by τ . The *mesh category* $K(\mathbb{Z}\mathbb{A}_n)$ is defined as the quotient category of the path category of $\mathbb{Z}\mathbb{A}_n$ by the *mesh ideal*. For any $r \in \mathbb{N}$, we denote by $K(\mathbb{Z}\mathbb{A}_n)/\tau^r$ the quotient category of $K(\mathbb{Z}\mathbb{A}_n)$ by the cyclic group $\tau^{r\mathbb{Z}}$.

Corollary 4.9. *Let $A = KQ/I$ be a simple polygon-tree algebra, where the gluing components of Q are Q_0, Q_1, \dots, Q_N . Then $D_{sg}^b(A)$ is triangle equivalent to $K(\mathbb{Z}\mathbb{A}_{m-3N+d_Q-2})/\tau^{m-3N+d_Q}$.*

Proof. From [Rie2], we get that the stable category $\underline{\text{mod}}(\mathcal{N}_{m-3N+d_Q})$ is equivalent to

$$K(\mathbb{Z}\mathbb{A}_{m-3N+d_Q-2})/\tau^{m-3N+d_Q}.$$

So the result follows from Theorem 4.4 immediately. \square

Example 4.10. *Let $A = KQ/I$ be the polygon-tree algebra with Q as Figure 9.1 shows. It is not simple, and also a cluster-tilted algebra of type \mathbb{E}_6 . From [CGL], we know that*

$$D_{sg}^b(A) \simeq \underline{\text{mod}}\mathcal{N}_4,$$

which does not satisfy the formula in Theorem 4.4.

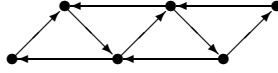


Figure 9.1. A non-simple polygon-tree quiver.

We have the following direct corollaries.

Corollary 4.11. *Let $A = KQ/I$ be a floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . Then*

$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m-3n+d_Q}).$$

Corollary 4.12 ([CGL]). *Let $A = KQ/I$ be a floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . If A is a cluster-tilted algebra of type \mathbb{D} , then*

$$D_{sg}^b(A) \simeq \underline{\text{mod}}(\mathcal{N}_{m_0+d_Q}).$$

Let $A = KQ/I$ be a quotient of the path algebra of the following quiver Q modulo the ideal I generated by the elements described below (in Q the arm with arrows x_i contains p_i arrows):

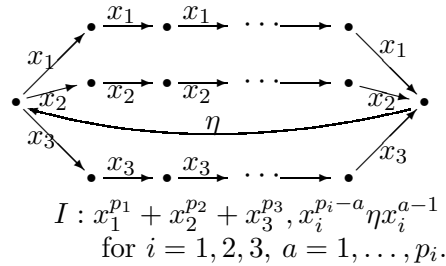


Figure 9.2. Cluster-tilted algebra of a canonical algebra with weights (p_1, p_2, p_3) .

From [BKL], we know that $A = KQ/I$ is a cluster-tilted algebra of a canonical algebra with weights (p_1, p_2, p_3) .

Corollary 4.13. *Let $A = KQ/I$ be the cluster-tilted algebra of a canonical algebra with weights (p_1, p_2, p_3) as above. If $p_1 = 2$, then $D_{sg}^b(A) \simeq \underline{\text{mod}}\mathcal{N}_{p_1+p_3-1}$.*

Proof. Since $p_1 = 2$, the quiver Q is as Figure 9.3 shows. Let e_{c_1} be the idempotent corresponding to the vertex c_1 . Let V be the quotient algebra of A modulo the ideal generated by e_{c_1} , namely $Ae_{c_1}A$. The quiver of V is obtained from Q by removing the vertex c_1 and the adjacent arrows α, β . It is easy to see that V is a floriated algebra.

Let $B = (1 - e_{c_1})A(1 - e_{c_1})$. Then $k\alpha$ and $k\beta$ are naturally left and right B -modules respectively since $\beta\eta = 0$ and $\eta\alpha = 0$ in B .

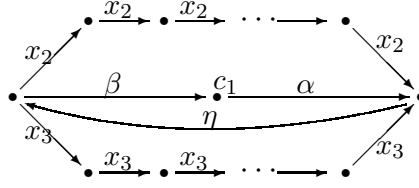


Figure 9.3. Cluster-tilted algebra of a canonical algebra with weights $(2, p_2, p_3)$.

We identify A with $\begin{pmatrix} B & k\alpha \\ k\beta & k \end{pmatrix}$, where the k in the southeast corner is identified with $e_{c_1} A e_{c_1}$, and $B = (1 - e_{c_1})A(1 - e_{c_1})$. The morphism $\phi : k\alpha \otimes_k k\beta \rightarrow B$ maps $\alpha \otimes \beta$ to $\alpha\beta$, so ϕ is a monomorphism. Furthermore, it is easy to see that $B/\text{Im } \phi = V$. Then Proposition 4.1 yields a triangulated equivalence $D_{sg}^b(A) \simeq D_{sg}^b(V)$. Using Theorem 4.4, we get that $D_{sg}^b(A) \simeq \underline{\text{mod}} \mathcal{N}_{p_1+p_3-1}$ immediately. \square

5. REPRESENTATION TYPE OF POLYGON-TREE ALGEBRAS

In this section, we study the representation type of polygon-tree algebras. We set $\mathbb{D}_3 = \mathbb{A}_3$ for convenience.

We are more concerned with the number of arrows between the vertices rather than arrows themselves, we assume that the quivers have no loops or oriented 2-cycles, but allow the arrows to be weighted by positive integers. If the weight of an arrow is 1, we do not specify it in the picture and call it a single arrow; if an arrow has weight 2 we call it a double arrow. For convenience, if an arrow $i \rightarrow j$ is weighted by a negative integer r , then it means that there is an arrow $j \rightarrow i$ with weight $-r$, and if an arrow $i \rightarrow j$ with weight 0, then it means that there is no arrow between i and j . By a subquiver of Q , we always mean a quiver obtained from Q by taking an induced (full) directed subgraph on a subset of vertices and keeping all its edge weights the same as in Q .

Definition 5.1 ([FZ]). *Let Q be a quiver without loops or oriented 2-cycles. The FZ-mutation of Q at a vertex k (denoted by $\mu_k^{FZ}(Q)$) is a new quiver Q^* , described as follows:*

1. Add a new vertex k^* .
2. If there is an arrow $i \rightarrow k$ with weight r , an arrow $k \rightarrow j$ with weight s and an arrow $j \rightarrow i$ with weight t in Q , then there is an arrow $j \rightarrow i$ with weight $t - rs$ in Q^* .
3. For any vertex i replace an arrow from i to k with an arrow from k^* to i , and replace an arrow from k to i with an arrow from i to k^* , with the same weights.
4. Remove the vertex k .

The FZ-mutation μ_k is involutive, so it defines a mutation-equivalence relation on quivers. A quiver Q is said to be of *finite mutation type* if its mutation-equivalence class is finite. It is well known that, in a finite mutation type quiver with at least three vertices, any edge is a single edge or a double edge. The most basic examples of finite mutation type quivers are Dynkin quivers and extended Dynkin quivers [BR].

Another important class of finite mutation type quivers has been obtained in [FoST] using a construction that associates quivers to certain triangulations of surfaces. In this paper, we will not use this construction, so we do not recall it here. We call these quivers that *come from the triangulation of a surface*.

Theorem 5.2 ([FeST]). *A connected quiver Q with at least three vertices is of finite mutation type if and only if it comes from the triangulation of a surface or it is mutation-equivalent to one of the exceptional types $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8, \mathbb{X}_6, \mathbb{X}_7$ (see Figure 10.2 and Figure 10.3).*

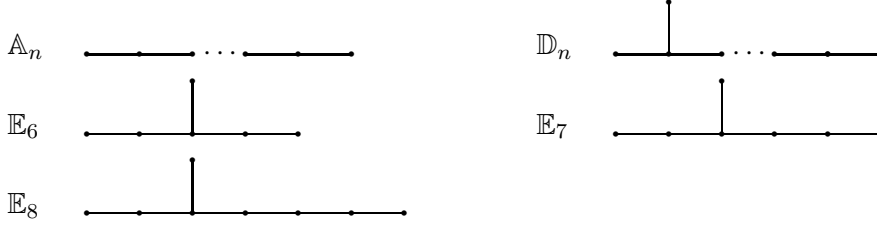


Figure 10.1. Dynkin quivers: the edges may be oriented arbitrarily.

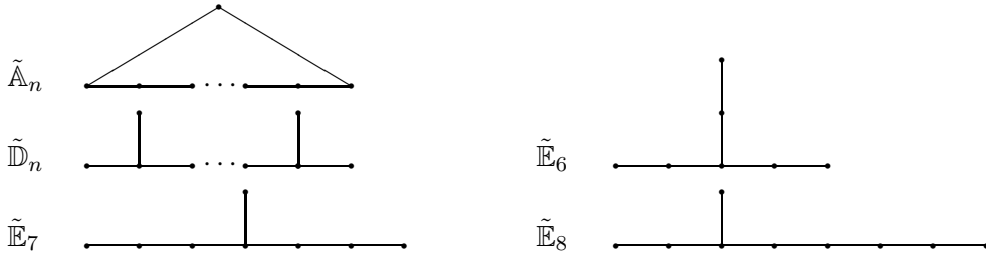


Figure 10.2. Extended Dynkin quivers: the edges may be oriented arbitrarily such that the quiver is acyclic.

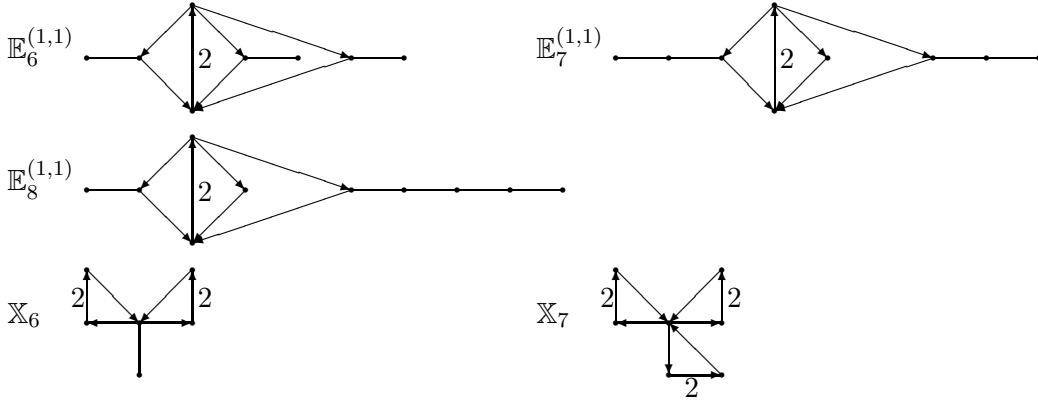


Figure 10.3. Exceptional quivers of finite mutation type which are acyclic: edges with unspecified orientation may be oriented arbitrarily.

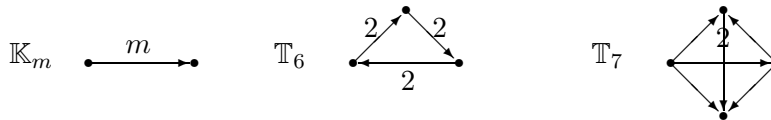


Figure 10.4. Quivers of finite mutation type with non-removable double arrows.

Theorem 5.3 ([S]). *Let Q be a connected quiver of finite mutation type. Suppose also that Q has a subquiver which is mutation-equivalent to \mathbb{E}_6 (resp. \mathbb{X}_6). Then any quiver which is mutation-equivalent to Q also contains a subquiver which is mutation-equivalent to \mathbb{E}_6 (resp. \mathbb{X}_6). Furthermore Q is mutation-equivalent to a quiver which is one of the types \mathbb{E} (resp. \mathbb{X}) given in Theorem 5.2.*

Lemma 5.4. *Let $A = KQ/I$ be a floriated algebra of $(Q_0, \{i_1, \dots, i_n\})$ by Q_1, \dots, Q_n . If Q is of finite mutation type, then we have one of the following statements:*

- (a) *A is a cluster-tilted algebra of type \mathbb{D} .*
- (b) *Q is mutation-equivalent to a quiver which is one of the types \mathbb{E} given in Theorem 5.2.*

Proof. Denote by (Q, W) the QP satisfying $A = J(Q, W)$. Denote by m_i the number of vertices in Q_i .

From Proposition 3.2, we get that (Q, W) is mutation-equivalent to (Q^1, W^1) as Figure 2.5 shows. Recall that Q^1 has only one oriented cycle c which has $m_0 + n$ vertices. We consider the following cases.

Case (1) If $m_0 + n = 3$, then $m_0 = 3$ and $n = 0$, which means that Q^1 is of type $\mathbb{D}_3 = \mathbb{A}_3$.

Case (2) If $m_0 + n = 4$, then $m_0 = 4$ and $n = 0$, or $m_0 = 3$ and $n = 1$. It is easy to see that it is of type \mathbb{D} .

Case (3) For $m_0 + n > 4$, if Q^1 is not a quiver of type \mathbb{D} , then we get that Q^1 is not an oriented cycle. So the quiver in Figure 11.1 is a subquiver of Q^1 , where the oriented cycle has $m_0 + n$ vertices. If $m_0 + n = 5$, then the quiver in Figure 11.1 is in the mutation-equivalence class of \mathbb{E}_6 , and the result follows from Theorem 5.3. If $m_0 + n > 5$, then the quiver in Figure 11.1 also admits a subquiver as Figure 11.2 shows, which is a quiver of type \mathbb{E}_6 . So Q^1 admits a subquiver mutation-equivalent to a quiver of type \mathbb{E}_6 , and then the result follows from Theorem 5.3.

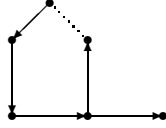


Figure 11.1. Subquiver of Q^1 .

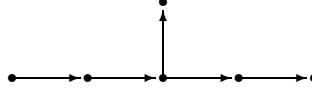


Figure 11.2. A quiver of type \mathbb{E}_6 . □

Lemma 5.5 ([GLS]). *Assume that Q is not mutation equivalent to one of the quivers $\mathbb{T}_1, \mathbb{T}_2, \mathbb{X}_6, \mathbb{X}_7$ or \mathbb{K}_m with $m \geq 3$. Then for any non-degenerate potential S on Q the following hold:*

- (a) $J(Q, S)$ is representation-finite if and only if Q is of type $\mathbb{A}_n, \mathbb{D}_n (n \geq 4), \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.
- (b) $J(Q, S)$ is tame if and only if Q is of finite mutation type.
- (c) $J(Q, S)$ is wild if and only if Q is of infinite mutation type.

Now we can get our main result of this section.

Theorem 5.6. *Let $A = KQ/I$ be a polygon-tree algebra, where the gluing components of Q are Q_0, Q_1, \dots, Q_N . Then*

- (a) A is of representation finite type if and only if A is in the mutation-equivalence class of type $\mathbb{A}_3, \mathbb{D}_n (n \geq 4), \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 ;
- (b) A is of tame representation type which is not representation finite if and only if A is in the mutation-equivalence class of type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8, \mathbb{E}_6^{(1,1)}, \mathbb{E}_7^{(1,1)}$ or $\mathbb{E}_8^{(1,1)}$. In particular, in this case, the number of vertices in Q is no more than 10.

Proof. Denote by m_i the number of vertices in Q_i , and $m = \sum_{i=0}^N m_i$. Then the number of vertices in Q is $m - 2N$. Since the polygon-tree quiver Q is not in the mutation-equivalence class of type \mathbb{A} when $m - 2N > 3$, (a) follows from Lemma 5.5 directly.

For (b), if A is of tame representation type which is not representation finite, then Q is of finite mutation type, and any floriated subquiver $Q(i)$ ($0 \leq i \leq N$) of Q is also of finite mutation type. If $Q(i)$ admits a subquiver in the mutation class of \mathbb{E}_6 , then Theorem 5.3 shows that Q is mutation-equivalent to a quiver which is one of the types \mathbb{E} , together with that A is not representation finite, we get that A is in the mutation-equivalence class of $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8, \mathbb{E}_6^{(1,1)}, \mathbb{E}_7^{(1,1)}$ or $\mathbb{E}_8^{(1,1)}$.

If each $Q(i)$ does not admit a subquiver in the mutation class of \mathbb{E}_6 , we get that $Q(i)$ is of type \mathbb{D} for any $0 \leq i \leq N$ by Lemma 5.4. If there exists Q_i such that $m_i \geq 4$ for some i , without loss of generality, we assume that $m_0 \geq 4$. We assume that the oriented chordless subquivers adjacent to Q_0 are Q_1, \dots, Q_n . Note that $n \geq 1$ since Q is not of type \mathbb{D} . Then $m_j = 3$ for

$1 \leq j \leq n$. Since Q is not of type \mathbb{D} , there exists Q_k with $1 \leq k \leq n$ such that it admits an adjacent oriented chordless subquiver different to Q_0 , and then $Q(k)$ is not of type \mathbb{D} , which is a contradiction. Thus $m_i = 3$ for any $0 \leq i \leq N$.

If each Q_i admits only one adjacent oriented chordless subquiver, then Q has only 2 gluing components Q_0, Q_1 . Since $m_i = 3$ for $i = 0, 1$, it is easy to see that Q is of type \mathbb{D} . So there exists a gluing component admitting at least 2 adjacent ones.

Without loss of generality, we assume that Q_1, Q_2 are adjacent gluing components of Q_0 . Since Q is not of type \mathbb{D} , then one of Q_1, Q_2 admits an adjacent oriented chordless subquiver different to Q_0 . We assume that Q_1 admits an adjacent oriented chordless subquiver Q_3 different to Q_0 . Then one of the two quivers in Figure 12 is a subquiver of Q . From [GP], we know that both of them are of type \mathbb{E}_6 . So Q is in the mutation-equivalence class of $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8, \mathbb{E}_6^{(1,1)}, \mathbb{E}_7^{(1,1)}$ or $\mathbb{E}_8^{(1,1)}$.



Figure 12. Quivers in the mutation-equivalence class of type \mathbb{E}_6 . □

Remark 5.7. In fact, from [GLS], we know that $J(Q, S)$ is wild if Q is mutation equivalent to one of the quivers $\mathbb{X}_6, \mathbb{X}_7$ and \mathbb{K}_m for $m \geq 3$, where S is a non-degenerate potential on Q . In the tame case, the quivers \mathbb{T}_i have no polygon-tree quivers in their mutation classes.

For floriated quivers of type $\mathbb{E}_6^{(1,1)}, \mathbb{E}_7^{(1,1)}, \mathbb{E}_8^{(1,1)}$, using Keller's applet for quiver mutations [Ke2], we can obtain that they are precisely as the following table shows. In particular, the floriated algebras of the following quivers are not cluster-tilted algebras.

classes	The floriated quivers				
$\mathbb{E}_7^{(1,1)}$					
$\mathbb{E}_8^{(1,1)}$					

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